The Intouch Triangle and the \( OI \)-line

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Abstract. We prove some interesting results relating the intouch triangle and the \( OI \) line of a triangle. We also give some interesting properties of the triangle center \( X_{57} \), the homothetic center of the intouch and excentral triangles.

1. Introduction

L. Emelyanov \([4]\) has recently given an interesting relation between the \( OI \)-line and the triangle of reflections of the intouch triangle. Here, \( O \) and \( I \) are respectively the circumcenter and incenter of the triangle. Given triangle \( ABC \) with intouch triangle \( XYZ \), let \( X_2, Y_2, Z_2 \) be the reflections of \( X, Y, Z \) in their respective opposite sides \( YZ, ZX, XY \). Then the lines \( AX_2, BY_2, CZ_2 \) intersect \( BC, CA, AB \) at the intercepts of the \( OI \)-line.

Emelyanov \([3]\) also noted that the intercepts of the points \( IX_2 \cap BC, IY_2 \cap CA, IZ_2 \cap AB \) form a triangle perspective with \( ABC \). See Figure 1. According to \([7]\), this perspector is the point

\[
X_{1442} = \left( \frac{a(b^2 + bc + c^2 - a^2)}{s-a} : \frac{b(c^2 + ca + a^2 - b^2)}{s-b} : \frac{c(a^2 + ab + b^2 - c^2)}{s-c} \right)
\]

Figure 1.
on the Soddy line joining the incenter and the Gergonne point.

In this paper we generalize these results. We work with barycentric coordinates with reference to triangle $ABC$.

2. The triangle center $X_{57}$

Let $a$, $b$, $c$ be the lengths of the sides $BC$, $CA$, $AB$ of triangle $ABC$, and $s = \frac{1}{2}(a + b + c)$ the semiperimeter. The intouch triangle $XYZ$ and the excentral triangle (with the excenters as vertices) are clearly homothetic, since their corresponding sides are perpendicular to the same angle bisector of triangle $ABC$. These triangles are respectively the cevian triangle of the Gergonne point \( \left( \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \right) \) and the anticevian triangle of the incenter \( (a : b : c) \), their homothetic center has coordinates

\[
(a(-a(s-a) + b(s-b) + c(s-c)) : \cdots : \cdots ) = (2a(s-b)(s-c) : \cdots : \cdots ) = \left( \frac{a}{s-a} : \cdots : \cdots \right).
\]

This is the triangle center $X_{57}$ in [6], defined as the isogonal conjugate of the Mittenpunkt $X_9 = (a(s-a) : b(s-b) : c(s-c))$. This is a point on the $OI$-line since the two triangles in question have circumcenters $I$ and $X_{40}$ (the reflection of $I$ in $O$), \(^1\)

We give some interesting properties of the triangle $X_{57}$.

Since $ABC$ is the orthic triangle of the excentral triangle, it is homothetic to the orthic triangle $X_1Y_1Z_1$ of $XYZ$ with the same homothetic center $X_{57}$. See Figure 2.

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\(^1\)The circumcircle of $ABC$ is the nine-point circle of the excentral triangle.
Let $DEF$ be the circumcevian triangle of the incenter $I$, and $D', E', F'$ the antipodes of $D, E, F$ in the circumcircle. In other words, $D$ and $D'$ are the midpoints of the two arcs $BC, D'$ on the arc containing the vertex $A$; similarly for the other two pairs. Clearly, 
\[ D = \left( \frac{a^2}{-(b+c)} : \frac{b^2}{b} : \frac{c^2}{c} \right) = (-a^2 : b(b+c) : c(b+c)). \]

Similarly, 
\[ E = (a(c+a) : -b^2 : c(c+a)) \quad \text{and} \quad F = (a(a+b) : b(a+b) : -c^2). \]

To compute the coordinates of $D, E', F'$, we make use of the following formula.

**Lemma 1.** Let $P = (a^2vw : b^2wu : c^2uv)$ be a point on the circumcircle (so that $u + v + w = 0$). For a point $Q = (x : y : z)$ different from $P$ and not lying on the circumcircle, the line $PQ$ intersects the circumcircle again at the point $(a^2vw + tx : b^2wu + ty : c^2uv + tz)$, where 
\[ t = \frac{b^2c^2u^2x + c^2a^2v^2y + a^2b^2w^2z}{a^2yz + b^2zx + c^2xy}. \] 

**(Proof.** Entering the coordinates 
\[ (X, Y, Z) = (a^2vw + tx : b^2wu + ty : c^2uv + tz) \]
into the equation of the circumcircle 
\[ a^2YZ + b^2ZX + c^2XY = 0, \]
we obtain 
\[ (a^2yz + b^2zx + c^2xy)t^2 + (b^2c^2u(v + w)x + c^2a^2v(w + u)y + a^2b^2w(u + v)z)t + a^2b^2c^2uvw(u + v + w) = 0. \]
Since $u + v + w = 0$, this gives $t = 0$ or the value given in (1) above. □

Let $M = (0 : 1 : 1)$ be the midpoint of $BC$. Applying Lemma 1 to $D$ and $M$, we obtain 
\[ D' = (-a^2 : b(b-c) : c(c-b)). \]

Similarly, 
\[ E' = (a(a-c) : -b^2 : c(c-a)) \quad \text{and} \quad F' = (a(a-b) : b(b-a) : -c^2). \]

Applying Lemma 1 to $D'$ and $X = (0 : a+b-c : c+a-b)$, (likewise to $E'$ and $Y$, and to $F'$ and $Z$), we obtain the points 
\[ X' = \left( \frac{-a^2}{a(b+c)-(b-c)^2} : \frac{b}{c+a-b} : \frac{c}{a+b-c} \right), \]
\[ Y' = \left( \frac{a}{b+c-a} : \frac{-b^2}{b(c+a)-(c-a)^2} : \frac{c}{a+b-c} \right), \]
\[ Z' = \left( \frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c^2}{c(a+b)-(a-b)^2} \right). \]
These are clearly the vertices of the circumcevian triangle of $X_{57}$. We summarize this in the following proposition.

**Proposition 2.** If $X'$ (respectively $Y'$, $Z'$) are the second intersections of $DX$ (respectively $EY$, $FZ$) and the circumcircle, then $X'Y'Z'$ is the circumcevian triangle of $X_{57}$.

![Figure 3](image)

**Remark.** The lines $D'X$, $E'Y$, $F'Z$ intersect at $X_{55}$, the internal center of similitude of the circumcircle and the incircle.

**Proposition 3.** Let $X''$, $Y''$, $Z''$ be the second intersections of the circumcircle with the lines $DX$, $EY$, $FZ$ respectively. The lines $AX''$, $BY''$, $CZ''$ bound the anticevian triangle of $X_{57}$.

**Proof.** By Lemma 1, these are the points

$$X'' = \left( \frac{a^2}{s-a} : \frac{b(b-c)}{s-b} : \frac{c(c-b)}{s-c} \right),$$

$$Y'' = \left( \frac{a(a-c)}{s-a} : \frac{b^2}{s-b} : \frac{c(c-a)}{s-c} \right),$$

$$Z'' = \left( \frac{a(a-b)}{s-a} : \frac{b(b-a)}{s-b} : \frac{c^2}{s-c} \right).$$

The lines $AX''$, $BY''$, $CZ''$ have equations

$$\frac{s-a}{a}x + \frac{s-b}{b}y + \frac{s-c}{c}z = 0,$$

$$\frac{s-a}{a}x + \frac{s-b}{b}y = 0.$$
Remark. The lines $DX$, $EY$, $FZ$ intersect at $X_{56}$, the external center of similitude of the circumcircle and incircle.

**Proposition 4.** $X_{57}$ is the perspector of the triangle bounded by the polars of $A$, $B$, $C$ with respect to the circle through the excenters.

**Proof.** As is easily verified, the equation of the circumcircle of the excentral triangle is

$$a^2yz + b^2zx + c^2xy + (x + y + z)(bcx + cay + abz) = 0.$$

The polars are the lines

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0,$$

$$\frac{x}{a} + \frac{y}{c} + \frac{z}{b} = 0,$$

$$\frac{x}{b} + \frac{y}{c} + \frac{z}{a} = 0.$$

They bound a triangle with vertices
\[
\left( -\frac{a(s^2-bc)}{s(s-b)(s-c)} : \frac{b}{s-b} : \frac{c}{s-c} \right), \\
\left( \frac{a}{s-a} : -\frac{b(s^2-ca)}{s(s-c)(s-a)} : \frac{c}{s-c} \right), \\
\left( \frac{a}{s-a} : \frac{s}{s-b} : -\frac{c(s^2-ab)}{s(s-a)(s-b)} \right),
\]

This clearly has perspector \( X_{57} \). \qed

**Proposition 5.** \( X_{57} \) is the perspector of the reflections of the Gergonne point in the intouch triangle.

![Figure 5](image)

More generally, the reflection triangle of \( P = (u : v : w) \) in the cevian triangle of \( P \) is perspective with \( ABC \) at

\[
\left( u \left( \frac{-a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2 + c^2 - a^2}{vw} \right) : \cdots : \cdots \right).
\]

See [2]. For example, if \( P \) is the incenter, this perspector is the point

\[
X_{35} = (a^2(b^2 + bc + c^2 - a^2) : b^2(c^2 + ca + a^2 - b^2) : c^2(a^2 + ab + b^2 - c^2))
\]

which divides the segment \( OI \) in the ratio \( OX_{35} : X_{35}I = R : 2r \).

Finally, we also mention from [5] that \( X_{57} \) is the orthocorrespondent of the incenter. This means that the trilinear polar of \( X_{57} \), namely, the line

\[
\frac{s-a}{a}x + \frac{s-b}{b}y + \frac{s-c}{c}z = 0
\]
intersects the sidelines $BC, CA, AB$ at $X, Y, Z$ respectively such that $IX \perp IA$, $IY \perp IB$, and $IZ \perp IC$.

3. A locus of perspectors

As an extension of the result of [4], we consider, for a real number $t$, the triangle $X_tY_tZ_t$ with $X_t, Y_t, Z_t$ dividing the segments $XX_1, YY_1, ZZ_1$ in the ratio

$$XX_t : X_tX_1 = YY_t : Y_tY_1 = ZZ_t : Z_tZ_1 = t : 1 - t.$$  

**Proposition 6.** The triangle $X_tY_tZ_t$ is perspective with $ABC$. The locus of the perspector is the Soddy line joining the incenter to the Gergonne point.

**Proof.** We compute the coordinates of $X_t, Y_t, Z_t$. It is well known that $BX = s - b, XC = s - c, etc.,$ so that, in absolute barycentric coordinates,

$$X = \frac{(s-c)B + (s-b)C}{a}, \quad Y = \frac{(s-a)C + (s-c)A}{b}, \quad Z = \frac{(s-b)A + (s-a)B}{c}.$$  

Since the intouch triangle $XYZ$ has (acute) angles $\frac{b+c}{2}, \frac{c+a}{2},$ and $\frac{A+B}{2}$ at $X, Y, Z$ respectively, the pedal $X_1$ of $X$ on $YZ$ divides the segment in the ratio

$$XY_1 : X_1Y = \cot \frac{C + A}{2} : \cot \frac{A + B}{2} = \tan \frac{B}{2} : \tan \frac{C}{2} = s - c : s - b.$$  

Similarly, $Y_1$ and $Z_1$ divide $ZX$ and $XY$ in the ratios

$$ZY_1 : Y_1X = s - a : s - c, \quad XZ_1 : Z_1Y = s - b : s - a.$$  

In absolute barycentric coordinates,

$$X_1 = \frac{(s-b)Y + (s-c)Z}{a} = \frac{(b+c)(s-b)(s-c)A + b(s-c)(s-a)B + c(s-a)(s-b)C}{abc}.$$  

It follows that

$$X_t = (1-t)X + tX_1 = \frac{t(b+c)(s-b)(s-c)A + b(s-c)(c-t(s-b))B + c(s-b)(b-t(s-c))C}{abc}.$$  

In homogeneous barycentric coordinates, this is

$$X_t = (t(b+c)(s-b)(s-c) : b(s-c)(c-t(s-b)) : c(s-b)(b-t(s-c)).$$  

The line $IX_t$ has equation

$$bc(b-c)(s-a)x + c(s-b)(ab - 2s(s-c)t)y - b(s-c)(ca - 2s(s-b)t)z = 0.$$  

The line $IX_t$ intersects $BC$ at the point

$$X_t' = (0 : b(s-c)(ca - 2s(s-b)t) : c(s-b)(ab - 2s(s-c)t)) = \left(0 : \frac{b(ca - 2s(s-b)t)}{s-b} : \frac{c(ab - 2s(s-c)t)}{s-c}\right).$$
Similarly, the lines $IY_t$ and $IZ_t$ intersect $CA$ and $AB$ respectively at

$$Y'_t = \left( \frac{a(bc - 2s(s - a)t)}{s - a}, \frac{c(ab - 2s(s - c)t)}{s - c} : 0 \right),$$

$$Z'_t = \left( \frac{a(bc - 2s(s - a)t)}{s - a}, \frac{b(ca - 2s(s - b)t)}{s - b} : 0 \right).$$

The triangle $X'_tY'_tZ'_t$ is perspective with $ABC$ at the point

$$\left( \frac{a(bc - 2s(s - a)t)}{s - a}, \frac{b(ca - 2s(s - b)t)}{s - b}, \frac{c(ab - 2s(s - c)t)}{s - c} \right).$$

As $t$ varies, this perspector traverses a straight line. Since the perspector is the Gergonne point for $t = 0$ and the incenter for $t = \infty$, this line is the Soddy line joining these two points.

The Soddy line has equation

$$(b - c)(s - a)^2 x + (c - a)(s - b)^2 y + (a - b)(s - c)^2 z = 0.$$  

Here are some triangle centers on the Soddy line, with the corresponding values of $t$. The symbol $r_a$ stands for the radius of the $A$-excircle.

| $t$ | perspector $| \begin{array}{c} \text{first barycentric coordinate} \\
1 & X_{77} & a(b^2 + c^2 - a^2) \\
2 & X_{1442} & a(b^2 + bc + c^2 - a^2) \\
\frac{1}{2} & X_{269} & (s-a)^2 \\
\frac{K}{s} & X_{481} & 2r_a - a \\
\frac{K}{s_a} & X_{482} & 2r_a + a \\
\frac{K}{s} & X_{175} & r_a - a \\
\frac{-2K}{s_a} & X_{176} & r_a + a \\
\frac{3K}{2s} & X_{1372} & 4r_a - 3a \\
\frac{-2K}{s_a} & X_{1371} & 4r_a + 3a \\
\frac{K}{s} & X_{1374} & 4r_a - a \\
\frac{-2K}{s} & X_{1373} & 4r_a + a \\
\end{array}$ |

The infinite point of the Soddy point is the point $X_{516} = (2a^3 - (b+c)(a^2 + (b-c)^2)) : 2b^3 - (c+a)(b^2 + (c-a)^2) : 2c^3 - (a+b)(c^2 + (a-b)^2)).$

It corresponds to $t = \frac{R(4R+r)}{s^2}$. The deLongchamps point $X_{20}$ also lies on the Soddy line. It corresponds to $t = \frac{2R(2R+r)}{s^2}$.

4. Emelyanov’s first problem

From the coordinates of $X_t$, we easily find the intersections

$$A_t = AX_t \cap BC, \quad B_t = BX_t \cap CA, \quad C_t = CX_t \cap AB.$$
These are
\[ A_t = (0 : b(s - c)(c - (s - b)t) : c(s - b)(b - (s - c)t)), \]
\[ B_t = (a(s - c)(c - (s - a)t) : c(s - a)(a - (s - c)t)), \]
\[ C_t = (a(s - b)(b - (s - a)t) : b(s - a)(a - (s - b)t) : 0). \] (2)

They are collinear if and only if
\[ (a - (s - b)t)(b - (s - c)t)(c - (s - a)t) + (a - (s - c)t)(b - (s - a)t)(c - (s - b)t) = 0. \] (3)

Since this is a cubic equation in \( t \), there are three values of \( t \) for which \( A_t, B_t, C_t \) are collinear. One of these is \( t = 2 \) according to [4]. The other two roots are given by
\[ abc - abct + 2(s - a)(s - b)(s - c)t^2 = 0. \] (4)

Since \( abc = 4Rrs \) and \( (s - a)(s - b)(s - c) = r^2s \), where \( R \) and \( r \) are respectively the circumradius and inradius, this becomes
\[ 2R - 2Rt + rt^2 = 0. \] (5)

From this,
\[ t = \frac{R \pm \sqrt{R^2 - 2Rr}}{r} = \frac{R \pm d}{r}, \]
where \( d \) is the distance between \( O \) and \( I \).

We identify the lines corresponding to these two values of \( t \).

**Proposition 7.** Corresponding to the two roots of (4), the lines containing \( A_t, B_t, C_t \) are the tangents to the incircle perpendicular to the \( OI \)-line.

**Lemma 8.** Consider a triangle \( ABC \) with intouch triangle \( XYZ \), and a line \( L \) intersecting the sides \( BC, CA, AB \) at \( A', B', C' \) respectively. The line \( L \) is tangent to the incircle if and only if one of the following conditions holds.

1. The intersection \( BB' \cap CC' \) lies on the line \( YZ \).
2. The intersection \( CC' \cap AA' \) lies on the line \( ZX \).
3. The intersection \( AA' \cap BB' \) lies on the line \( XY \).

**Proof.** Let \( A'B' \) be a tangent to the incircle. By Brianchon’s theorem applied to the circumscribed hexagon \( AY'B'A'XB \) it immediately follows that \( AA', YX \) and \( B'B \) are concurrent.

Now suppose \( AA', YX \) and \( B'B \) are concurrent. Consider the tangent through \( A' \) (different from \( BC \)) to the incircle. Let \( B'' \) be the intersection of this tangent with \( AC \). It follows from the preceeding that \( AA', YX \) and \( B''B \) are concurrent. Therefore \( B'' \) must coincide with \( B' \). This means that \( A'B' \) is a tangent to the incircle. \( \square \)
5. Proof of Proposition 7

The lines $BB_t$ and $CC_t$ intersect at the point

$$A'' = \left( \frac{a}{s-a}(b - (s-a)t)(c - (s-a)t) : \frac{b}{s-b}(c - (s-a)t)(a - (s-b)t) : \frac{c}{s-c}(a - (s-c)t)(b - (s-a)t) \right).$$

This point lies on the line $YZ : -(s-a)x + (s-b)y + (s-c)z = 0$ if and only if

$$-a(b - (s-a)t)(c - (s-a)t) + b(c - (s-a)t)(a - (s-b)t) + c(a - (s-c)t)(b - (s-a)t) = 0.$$

This reduces to equation (4) above. By Lemma 8, these two lines are tangent to the incircle. We claim that these are the tangents perpendicular to the line $OI$. From the coordinates given in (2), the equation of the line $B_tC_t$ is

$$-a(s-a)(a - (s-b)t)(a - (s-c)t)x + \frac{a}{s-b}(a - (s-c)t)(b - (s-a)t)y + \frac{a}{s-c}(a - (s-b)t)(c - (s-a)t)z = 0.$$

According to [6], lines perpendicular to $OI$ have infinite point

$$X_{513} = (a(b-c) : b(c-a) : c(a-b)).$$

The line $B_tC_t$ contains the infinite point $X_{513}$ if and only if the same equation (4) holds. This shows that the two lines in question are indeed the tangents to the incircle perpendicular to the $OI$-line.

References


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