Triangle-Conic Porism

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Abstract. We investigate, for a given triangle, inscribed triangles whose sides are tangent to a given conic.

Consider a triangle $A_1B_1C_1$ inscribed in $ABC$, and a conic $C$ inscribed in $A_1B_1C_1$. We ask whether there are other inscribed triangles in $ABC$ and tritangent to $C$. Restricting to circles, Ton Lecluse wrote about this problem in [6]. See also [5]. He suggested after use of dynamic geometry software that in general there is a second triangle tritangent to $C$ and inscribed in $ABC$. In this paper we answer Lecluse’s question.

Proposition 1. Let $A'B'C'$ be a variable triangle of which $B'$ and $C'$ lie on $CA$ and $AB$ respectively. If the sidelines of triangle $A'B'C'$ are tangent to a conic $C$, then the locus of $A'$ is either a conic or a line.

Proof. Let $XYZ$ be the points on $C$ and where $C'A'$, $A'B'$, and $B'C'$ respectively meet $C$. $ZX$ is the polar (with respect to $C$) of $B'$, which passes through a fixed point $P_B$, the pole of $CA$. Similarly $XY$ passes through a fixed point $P_C$. The mappings $Y \mapsto X$ and $X \mapsto Z$ are thus involutions on $C$. Hence $Y \mapsto Z$ is a projectivity. That means that the lines $YZ$ form a pencil of lines or envelope a conic according as $Y \mapsto Z$ is an involution or not. Consequently the poles of these lines, the points $A'$, run through a line $\ell_A$ or a conic $C_A$. □

Two degenerate triangles $A'B'C'$, corresponding to the tangents from $A$, arise as limit cases. Hence, when $Y \mapsto Z$ is an involution, the points $U_1$ and $U_2$ of contact of tangents from $A$ to $C$ are its fixed points, and $\ell_A = U_1U_2$ is the polar of $A$.

The conics $C$ and $C_A$ are tangent to each other in $U_1$ and $U_2$. We see that $C$ and $C_A$ generate a pencil, of which the pair of common tangents, and the polar of $A$ (as double line) are the degenerate elements. In view of this we may consider the line $\ell_A$ as a conic $C_A$ degenerated into a “double” line. We do so in the rest of this paper.

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Proposition 1 shows us that if there is one inscribed triangle tritangent to $C$, there will be in general another such triangle. This answers Lecluse’s question for the general case. But it turns out that the other cases lead to interesting configurations as well.

The number of intersections of $C_A$ with $BC$ gives the number of inscribed triangles tritangent to $C$. There may be infinitely many, if $C_A$ degenerates and contains $BC$. This implies that $BC = \ell_A$. By symmetry it is necessary that $ABC$ is self-polar with respect to $C$. Of course this applies also when the above $A$ runs through $\ell_A$ in the plane of the triangle bounded by $AB, CA$ and $\ell_A$.

There are two possibilities for $C_A$ and $BC$ to intersect in one “double” point. One is that $C_A$ is nondegenerate and tangent to $BC$. In this case, by reasons of continuity, the point of tangency belongs to one triangle $AB'C'$, and similar conics $C_B$ and $C_C$ are tangent to the corresponding side as well. The points of tangency form the cevian triangle of the perspector of $C$.

This can be seen by considering the point $M$ where $U_1U_2$ meets $BC$. The polar of $M$ with respect to $C$ passes through the pole of $U_1U_2$, and through the intersections of the polars of $B$ and $C$, hence the pole of $BC$. So the polar $\ell_M$ of $M$ is the $A$-cevian of the perspector\footnote{By Chasles’ theorem on polarity [1, 5.61], each triangle is perspective to its polar triangle. The perspector is called the perspector of the conic.} of $C$. The point where $U_1U_2$ and $\ell_M$ meet is the harmonic conjugate of $M$ with respect to $U_1$ and $U_2$. This all applies to $C_A$ as well. In case $C_A$ is tangent to $BC$, the point of tangency is the pole of $BC$, and is thus the trace of the perspector of $C$.

For example, if $C$ is the incircle of the medial triangle, the conic $C_A$ is tangent to $BC$ at its midpoint, and contains the points $(s : s - b : b), (s : c : s - c)$.
Consequently the anti-cevian triangle of $R$ by duality.

But that means that each point on $\ell$ cannot be cevian triangles.

In the case $ABC$ is selfpolar with respect to $K$. We conclude that two distinct triangles inscribed in $ABC$ and circumscribing $C$ cannot be cevian triangles.

The other possibility for a double point is when $C_A$ degenerates into $\ell_A$. To investigate this case we prove the following proposition.

**Proposition 2.** If $C_A$ degenerates into a line, the triangle $ABC$ is selfpolar with respect to each conic tangent to the sides of two cevian triangles. The cevian triangle of the trilinear pole of any tangent to such a conic is tritangent to this conic.

**Proof.** Let $P$ be a point and $APBP'C'$ its anticevian triangle. $ABC$ is a polar triangle with respect to each conic through $APBP'C'$, as $ABC$ are the diagonal points of the complete quadrilateral $PA^PBPC'$. Now consider a second anticevian triangle $A^QBP'C^Q$ of $Q$. The vertices of $APBP'C'$ and $A^QBP'C^Q$ lie on a conic of $K$. But we also know that triangle $PBPC'$ is the anticevian triangle of $A'$. So $PBPC'$ and $A^QBP'C^Q$ lie on a conic as well, and having 5 common points this must be $C$. We conclude that $ABC$ is selfpolar with respect to $K$.

Let $R$ be a point on $K$. $AR$ intersects $K$ in a second point $R'$. Let $R_A$ be the intersection $AR$ and $BC$, then $R$ and $R'$ are harmonic with respect to $A$ and $R_A$. But that means that $R' = A^R$ is the $A-$vertex of the anti-cevian triangle of $R$. Consequently the anti-cevian triangle of $R$ lies on $K$. Proposition 2 is now proved by duality.

In the proof $BP'C'$ is the side of two anticevian triangles inscribed in $K$ - by duality this means that the vertex of a cevian triangle tangent to $K$ is a common vertex of two such cevian triangles. In the case of $\ell_A$ intersecting $BC$ in a double point, clearly the two triangles are cevian triangles with respect to the triangle bounded by $AB$, $AC$ and $\ell_A$. Were they cevian triangles also with respect to $ABC$, then the four sidelines of these cevian triangles would form the dual of an anticevian triangle, and $ABC$ would be selfpolar with respect to $C$, and $\ell_A$ would be $BC$.

We conclude that two distinct triangles inscribed in $ABC$ and circumscribing $C$ cannot be cevian triangles.

In the case $ABC$ is selfpolar with respect to $C$, so that $C_A$ degenerates into $\ell_A$, not each point on $\ell_A$ belongs to (real) cevian triangles. On the other hand clearly infinitely many points on $\ell_A$ will lead to two cevian triangles tritangent to $C$. The persectors run through a quartic, the tripoles of the tangents to $C$.

**Theorem 3.** Given a triangle $ABC$ and a conic $C$, the triangle-conic poristic triangles inscribed in $ABC$ and tritangent to $C$ are as follows.

(i) There are be no triangle-conic poristic triangle.

(ii) $C$ is a conic inscribed in a cevian triangle, and $ABC$ is not self-polar with respect to $C$. In this case the cevian triangle is the only triangle-conic poristic triangle.

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2This follows from the dual of the well known theorem that two cevian triangles are circumscribed by and inscribed in a conic.
(iii) \( ABC \) is self-polar with respect to \( C \). In this case there are infinitely many triangle-conic poristic triangles.

(iv) There are two distinct triangle-conic poristic triangles, which are not cevian triangles.

Remarks. (1) In case of a conic with respect to which \( ABC \) is self-polar, instead of cevian triangles tritangent to \( C \), we should speak of cevian fourlines quadritangent to \( C \).

(2) When we investigate triangles inscribed in a conic and circumscribed to \( ABC \) we get similar results as Theorem 3, simply by duality.

In case \( C \) is a conic with respect to which \( ABC \) is self-polar, we see that each tangent to \( C \) belongs to two cevian triangles tritangent to \( C \) and that each point on \( C \) belongs to two anticevian triangles inscribed in \( C \). In this case speak of triangle-conic porism and conic-triangle porism in extension of the well known Poncelet porism.

As an example, we consider the nine-point circle triangles, hence the medial and orthic triangles. We know that these circumscribe a conic \( C_N \), with respect to which \( ABC \) is self-polar. By Proposition 2 we know that the perspectrices of the medial and orthic triangles are tangent to \( C_N \) as well, hence \( C_N \) must be a parabola tangent to the orthic axis. The barycentric equation of this parabola is

\[
\frac{x^2}{a^2(b^2-c^2)} + \frac{y^2}{b^2(c^2-a^2)} + \frac{z^2}{c^2(a^2-b^2)} = 0.
\]

Its focus is \( X_{115} \), its directrix the Brocard axis, and its axis is the Simson line of \( X_{98} \). See Figure 2. The parabola contains the infinite point \( X_{512} \) and passes through \( X_{661}, X_{647} \) and \( X_{2519} \). The Brianchon point of the parabola with respect to the medial triangle is \( X_{670}(\text{medial}) \).

The perspectors of the tangent cevian triangles run through the quartic

\[
a^2(b^2-c^2)y^2z^2 + b^2(c^2-a^2)z^2x^2 + c^2(a^2-b^2)x^2y^2 = 0,
\]

which is the isotomic conjugate of the conic

\[
a^2(b^2-c^2)x^2 + b^2(c^2-a^2)y^2 + c^2(a^2-b^2)z^2 = 0
\]

through the vertices of the antimedial triangle, the centroid, and the isotomic conjugates of the incenter and the orthocenter.

This special case leads us to amusing consequences, to which we were pointed by [2]. The sides of every cevian triangle and its perspectrix are tangent to one parabola inscribed in the medial triangle. Consequently the isotomic conjugates with respect to to the medial triangle of these are parallel.

In the dual case, we conclude for instance that the isotomic conjugates with respect to the antimedial triangle of the vertices and perspector \( D \) of any anticevian triangle are collinear with the centroid \( G \). The line is \( GD' \), where \( D' \) is the barycentric square of \( D \).

\[\text{3The isotomic conjugate of a line } \ell \text{ with respect to a triangle is the line passing through the intercepts of } \ell \text{ with the sides reflected through the corresponding midpoints. In [3] this is referred to as isotomic transversal.}\]
References


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