A Note on the Barycentric Square Roots of Kiepert Perspectors

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Abstract. Let $P$ be an interior point of a given triangle $ABC$. We prove that the orthocenter of the cevian triangle of the barycentric square root of $P$ lies on the Euler line of $ABC$ if and only if $P$ lies on the Kiepert hyperbola.

1. Introduction

In a recent Mathlinks message, the present author proposed the following problem.

Theorem 1. Given an acute triangle $ABC$ with orthocenter $H$, the orthocenter $H'$ of the cevian triangle of $\sqrt{H}$, the barycentric square root of $H$, lies on the Euler line of triangle $ABC$.

Paul Yiu has subsequently discovered the following generalization.

Theorem 2. The locus of point $P$ for which the orthocenter of the cevian triangle of the barycentric square root $\sqrt{P}$ lies on the Euler line is the part of the Kiepert hyperbola which lies inside triangle $ABC$.


The author is grateful to Professor Yiu for his generalization of the problem and his help in the preparation of this paper.
The barycentric square root is defined only for interior points. This is the reason why we restrict to acute angled triangles in Theorem 1 and to the interior points on the Kiepert hyperola in Theorem 2. It is enough to prove Theorem 2.

2. Trilinear polars

Let $A'B'C'$ be the cevian triangle of $P$, and $A_1, B_1, C_1$ be respectively the intersections of $B'C'$ and $BC$, $C'A'$ and $CA$, $A'B'$ and $AB$. By Desargues’ theorem, the three points $A_1, B_1, C_1$ lie on a line $\ell_P$, the trilinear polar of $P$.

![Figure 2](image.png)

If $P$ has homogeneous barycentric coordinates $(u : v : w)$, then the trilinear polar is the line

$$\ell_P : \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$  

For the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$, the trilinear polar

$$\ell_H : S_Ax + S_By + S_Cz = 0.$$  

is also called the orthic axis.

**Proposition 3.** The orthic axis is perpendicular to the Euler line.

This proposition is very well known. It follows easily, for example, from the fact that the orthic axis $\ell_H$ is the radical axis of the circumcircle and the nine-point circle. See, for example, [2, §§5.4,5].

The trilinear polar $\ell_P$ and the orthic axis $\ell_H$ intersect at the point

$$(u(S_Bv - S_Cw) : v(S_Cw - S_Au) : w(S_Au - S_Bv)).$$

In particular, $\ell_P$ and $\ell_H$ are parallel, i.e., their intersection is a point at infinity if and only if

$$u(S_Bv - S_Cw) + v(S_Cw - S_Au) + w(S_Au - S_Bv) = 0.$$  

Equivalently,

$$ (S_B - S_C)vw + (S_C - S_A)wu + (S_A - S_B)uw = 0. \quad (1)$$
Note that this equation defines the Kiepert hyperbola. Points on the Kiepert hyperbola are called Kiepert perspectors.

**Proposition 4.** The trilinear polar \( \ell_P \) is parallel to the orthic axis if and only if \( P \) is a Kiepert perspector.

3. The barycentric square root of a point

Let \( P \) be a point inside triangle \( ABC \), with homogeneous barycentric coordinates \((u : v : w)\). We may assume \( u, v, w > 0 \), and define the barycentric square root of \( P \) to be the point \( \sqrt{P} \) with barycentric coordinates \((\sqrt{u} : \sqrt{v} : \sqrt{w})\).

Paul Yiu [2] has given the following construction of \( \sqrt{P} \):

1. Construct the circle \( C_A \) with \( BC \) as diameter.
2. Construct the perpendicular to \( BC \) at the trace \( A' \) of \( P \) to intersect \( C_A \) at \( X' \).
3. Construct the bisector of angle \( BX'C \) to intersect \( BC \) at \( X \).

Then \( X \) is the trace of \( \sqrt{P} \) on \( BC \). Similar constructions on the other two sides give the traces \( Y \) and \( Z \) of \( \sqrt{P} \) on \( CA \) and \( AB \) respectively. The barycentric square root \( \sqrt{P} \) is the common point of \( AX, BY, CZ \).

**Proposition 5.** If the trilinear polar \( \ell_P \) intersects \( BC \) at \( A_1 \), then

\[
A_1X^2 = A_1B \cdot A_1C.
\]

**Proof.** Let \( M \) is the midpoint of \( BC \). Since \( A_1, A' \) divide \( B, C \) harmonically, we have \( MB^2 = MC^2 = MA_1 \cdot MA' \) (Newton’s theorem). Thus, \( MX'^2 = MA_1 \cdot MA' \). It follows that triangles \( MX'A_1 \) and \( MA'X' \) are similar, and \( \angle MX'A_1 = \angle MA'X' = 90^\circ \). This means that \( A_1X' \) is tangent at \( X' \) to the circle with diameter \( BC \). Hence, \( A_1X'^2 = A_1B \cdot A_1C \).

![Figure 3.](image-url)
To complete the proof it is enough to show that $A_1X = A_1X'$, i.e., triangle $A_1XX'$ is isosceles. This follows easily from
\[
\angle A_1X'X = \angle A_1X'B + \angle BX'X = \angle X'CB + \angle XX'C = \angle X'XA_1.
\]

\[\square\]

**Corollary 6.** If $X_1$ is the intersection of $YZ$ and $BC$, then $A_1$ is the midpoint of $XX_1$.

**Proof.** If $X_1$ is the intersection of $YZ$ and $BC$, then $X, X_1$ divide $B, C$ harmonically. The circle through $X, X_1$, and with center on $BC$ is orthogonal to the circle $C_A$. By Proposition 5, this has center $A_1$, which is therefore the midpoint of $XX_1$. \[\square\]

4. Proof of Theorem 2

Let $P$ be an interior point of triangle $ABC$, and $XYZ$ the cevian triangle of its barycentric square root $\sqrt{P}$.

**Proposition 7.** If $H'$ is the orthocenter of $XYZ$, then the line $OH'$ is perpendicular to the trilinear polar $\ell_P$.

**Proof.** Consider the orthic triangle $DEF$ of $XYZ$. Since $DEXY$, $EFYZ$, and $FDZX$ are cyclic, and the common chords $DX, EY, FZ$ intersect at $H'$, $H'$ is the radical center of the three circles, and
\[
H'D \cdot H'X = H'E \cdot H'Y = H'F \cdot H'Z.
\]

Consider the circles $\xi_A, \xi_B, \xi_C$, with diameters $XX_1, YY_1, ZZ_1$. These three circles are coaxial; they are the generalized Apollonian circles of the point $\sqrt{P}$. See [3]. As shown in the previous section, their centers are the points $A_1, B_1, C_1$ on the trilinear polar $\ell_P$. See Figure 4.

Now, since $D, E, F$ lie on the circles $\xi_A, \xi_B, \xi_C$ respectively, it follows from (2) that $H'$ has equal powers with respect to the three circles. It is therefore on the radical axis of the three circles.

We show that the circumcenter $O$ of triangle $ABC$ also has the same power with respect to these circles. Indeed, the power of $O$ with respect to the circle $\xi_A$ is
\[
A_1O^2 - A_1X^2 = OA_1^2 - R^2 - A_1X^2 + R^2 = A_1B \cdot A_1C - A_1X^2 + R^2 = R^2
\]
by Proposition 5. The same is true for the circles $\xi_B$ and $\xi_C$. Therefore, $O$ also lies on the radical axis of the three circles. It follows that the line $OH'$ is the radical axis of the three circles, and is perpendicular to the line $\ell_P$ which contains their centers. \[\square\]

The orthocenter $H'$ of $XYZ$ lies on the Euler line of triangle $ABC$ if and only if the trilinear polar $\ell_P$ is parallel to the Euler line, and hence parallel to the orthic axis by Proposition 3. By Proposition 4, this is the case precisely when $P$ lies on the Kiepert hyperbola. This completes the proof of Theorem 2.
Theorem 8. The orthocenter of the cevian triangle of $\sqrt{P}$ lies on the Brocard axis if and only if $P$ is an interior point on the Jerabek hyperbola.

Proof. The Brocard axis $OK$ is orthogonal to the Lemoine axis. The locus of points whose trilinear polars are parallel to the Brocard axis is the Jerabek hyperbola.

5. Coordinates

In homogeneous barycentric coordinates, the orthocenter of the cevian triangle of $(u : v : w)$ is the point
\[
\left( S_B \left( \frac{1}{w} + \frac{1}{u} \right) + S_C \left( \frac{1}{u} + \frac{1}{v} \right) \right) \left( -S_A \left( \frac{1}{u} + \frac{1}{w} \right)^2 + S_B \left( \frac{1}{w^2} - \frac{1}{w^2} \right) + S_C \left( \frac{1}{u^2} - \frac{1}{v^2} \right) \right) \\
\vdots \; \vdots \; \vdots
\]

Applying this to the square root of the orthocenter, with \((u^2 : v^2 : w^2) = \left( \frac{1}{S_A^2} : \frac{1}{S_B^2} : \frac{1}{S_C^2} \right)\), we obtain
\[
\left( a^2 S_A : \sqrt{S_{ABC}} + S_{BC} \sum_{cyclic} a^2 \sqrt{S_A : \cdots : \cdots} \right),
\]
which is the point \(H'\) in Theorem 1.

More generally, if \(P\) is the Kiepert perspector
\[
K(\theta) = \left( \frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta} \right),
\]
the orthocenter of the cevian triangle of \(\sqrt{P}\) is the point
\[
\left( a^2 S_A \sqrt{(S_A + S_\theta)(S_B + S_\theta)(S_C + S_\theta)} \right) \\
+ S_{BC} \sum_{cyclic} a^2 \sqrt{S_A + S_\theta} + a^2 S_\theta \sum_{cyclic} S_A \sqrt{S_A + S_\theta} : \cdots : \cdots
\]

References

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