The Edge-Tangent Sphere of a Circumscriptible Tetrahedron

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Abstract. A tetrahedron is circumscriptible if there is a sphere tangent to each of its six edges. We prove that the radius \( \ell \) of the edge-tangent sphere is at least \( \sqrt{3} \) times the radius of its inscribed sphere. This settles affirmatively a problem posed by Z. C. Lin and H. F. Zhu. We also briefly examine the generalization into higher dimension, and pose an analogous problem for an \( n \)-dimensional simplex admitting a sphere tangent to each of its edges.

1. Introduction

Every tetrahedron has a circumscribed sphere passing through its four vertices and an inscribed sphere tangent to each of its four faces. A tetrahedron is said to be circumscriptible if there is a sphere tangent to each of its six edges (see [1, §§786–794]). We call this the edge-tangent sphere of the tetrahedron.

Let \( \mathcal{P} \) denote a tetrahedron \( P_0P_1P_2P_3 \) with edge lengths \( P_iP_j = a_{ij} \) for \( 0 \leq i < j \leq 3 \). The following necessary and sufficient condition for a tetrahedron to admit an edge-tangent sphere can be found in [1, §§787, 790, 792]. See also [4, 6].

**Theorem 1.** The following statement for a tetrahedron \( \mathcal{P} \) are equivalent.

(1) \( \mathcal{P} \) has an edge-tangent sphere.
(2) \( a_{01} + a_{23} = a_{02} + a_{13} = a_{03} + a_{12} \);
(3) There exist \( x_i > 0, i = 0, 1, 2, 3 \), such that \( a_{ij} = x_i + x_j \) for \( 0 \leq i < j \leq 3 \).

For \( i = 0, 1, 2, 3 \), \( x_i \) is the length of a tangent from \( P_i \) to the edge-tangent sphere of \( \mathcal{P} \). Let \( \ell \) denote the radius of this sphere.

**Theorem 2.** [1, §793] The radius of the edge-tangent sphere of a circumscriptible tetrahedron of volume \( V \) is given by

\[
\ell = \frac{2x_0x_1x_2x_3}{3V}.
\]

Lin and Zhu [4] have given the formula (1) in the form

\[
\ell^2 = \frac{(2x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum_{0 \leq i < j \leq 3} x_i x_j - (x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_0^2 + x_0^2x_1^2 + x_0^2x_1^2 + x_1^2x_2^2)}.
\]

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The fact that this latter denominator is \((3V)^2\) follows from the formula for the volume of a tetrahedron in terms of its edges:

\[
V^2 = \frac{1}{288} \begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & (x_0 + x_1)^2 & (x_0 + x_2)^2 \\
1 & (x_0 + x_1)^2 & 0 & (x_0 + x_3)^2 \\
1 & (x_0 + x_2)^2 & (x_1 + x_2)^2 & 0 \\
1 & (x_0 + x_3)^2 & (x_1 + x_3)^2 & (x_2 + x_3)^2
\end{vmatrix}.
\]

Lin and Zhu \textit{op. cit.} obtained several inequalities for the edge-tangent sphere of \(\mathcal{P}\). They also posed the problem of proving or disproving \(\ell \geq 3r^2\) for a circumscribable tetrahedron. See also [2]. The main purpose of this paper is to settle this problem affirmatively.

\textbf{Theorem 3.} For a circumscribable tetrahedron with inradius \(r\) and edge-tangent sphere of radius \(\ell\), \(\ell \geq \sqrt{3}r\).

\section{Two inequalities}

\textbf{Lemma 4.} If \(x_i > 0\) for \(0 \leq i \leq 3\), then

\[
\left(\frac{x_1 + x_2 + x_3}{x_1x_2x_3} + \frac{x_2 + x_3 + x_0}{x_2x_3x_0} + \frac{x_3 + x_0 + x_1}{x_3x_0x_1} + \frac{x_0 + x_1 + x_2}{x_0x_1x_2}\right) 4(x_0x_1x_2x_3)^2 \geq 6. \tag{3}
\]

\textit{Proof.} From

\[
x_0^2x_1^2(x_2 - x_3)^2 + x_0^2x_2^2(x_1 - x_3)^2 + x_0^2x_3^2(x_1 - x_2)^2 + x_1^2x_2^2(x_0 - x_3)^2 + x_1^2x_3^2(x_0 - x_2)^2 + x_2^2x_3^2(x_0 - x_1)^2 \geq 0,
\]

we have

\[
x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2 \geq \frac{2}{3}x_0x_1x_2x_3 \sum_{0 \leq i < j \leq 3} x_ix_j,
\]

and

\[
2x_0x_1x_2x_3 \sum_{0 \leq i < j \leq 3} x_ix_j - \left(\frac{2}{3}x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2\right) \geq \frac{4}{3}x_0x_1x_2x_3 \sum_{0 \leq i < j \leq 3} x_ix_j,
\]

or

\[
\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3} \sum_{0 \leq i < j \leq 3} x_ix_j - \left(\frac{2}{3}x_0^2x_1^2x_2^2 + x_0^2x_2^2x_3^2 + x_0^2x_3^2x_1^2 + x_0^2x_1^2x_2^2\right) \geq \frac{4}{3}x_0x_1x_2x_3 \sum_{0 \leq i < j \leq 3} x_ix_j. \tag{4}
\]
On the other hand, it is easy to see that
\[
\frac{x_1 + x_2 + x_3}{x_1 x_2 x_3} + \frac{x_2 + x_3 + x_0}{x_2 x_3 x_0} + \frac{x_3 + x_0 + x_1}{x_3 x_0 x_1} + \frac{x_0 + x_1 + x_2}{x_0 x_1 x_2} = \frac{2}{x_0 x_1 x_2 x_3} \sum_{0 \leq i < j \leq 3} x_i x_j.
\]  
(5)

Inequality (3) follows immediately from (4) and (5).

\[\square\]

**Corollary 5.** For a circumscribable tetrahedron \( \mathcal{P} \) with an edge-tangent sphere of radius \( \ell \), and faces with inradii \( r_0, r_1, r_2, r_3 \),
\[
\left( \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) \ell^2 \geq 6.
\]
Equality holds if and only if \( \mathcal{P} \) is a regular tetrahedron.

**Proof.** From the famous Heron formula, the inradius of a triangle \( ABC \) of side-lengths \( a = y + z, b = z + x \) and \( c = x + y \) is given by
\[
r^2 = \frac{xyz}{x + y + z}.
\]
Applying this to the four faces of \( \mathcal{P} \), we see that the first factor on the left hand side of (3) is \( \left( \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) \). Now the result follows from (2).

\[\square\]

**Proposition 6.** Let \( \mathcal{P} \) be a circumscribable tetrahedron of volume \( V \). If, for \( i = 0, 1, 2, 3 \), the opposite face of vertex \( P_i \) has area \( \triangle_i \) and inradius \( r_i \), then
\[
(\triangle_0 + \triangle_1 + \triangle_2 + \triangle_3)^2 \geq \frac{9 V^2}{2} \left( \frac{1}{r_0^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} \right).
\]  
(6)
Equality holds if and only if \( \mathcal{P} \) is a regular tetrahedron.

![Figure 1.](image-url)
Proof. Let $\alpha$ be the angle between the planes $P_0 P_2 P_3$ and $P_1 P_2 P_3$. If the perpen-
diculares from $P_0$ to the line $P_2 P_3$ and to the plane $P_1 P_2 P_3$ intersect these at $Q_1$ and $H$ respectively, then $\angle P_0 Q H = \alpha$. See Figure 1. Similarly, we have the angles $eta$ between the planes $P_0 P_3 P_1$ and $P_1 P_2 P_3$, and $\gamma$ between $P_0 P_1 P_2$ and $P_1 P_2 P_3$. Note that

$$P_0 H = P_0 Q_1 \cdot \sin \alpha = P_0 Q_2 \cdot \sin \beta = P_0 Q_3 \cdot \sin \gamma.$$  

Hence,

$$P_0 H \cdot P_2 P_3 = 2 \triangle_1 \sin \alpha = 2 \sqrt{(\triangle_1 + \triangle_1 \cos \alpha)(\triangle_1 - \triangle_1 \cos \alpha)}, \quad (7)$$

$$P_0 H \cdot P_3 P_1 = 2 \triangle_2 \sin \beta = 2 \sqrt{(\triangle_2 + \triangle_2 \cos \beta)(\triangle_2 - \triangle_2 \cos \beta)}, \quad (8)$$

$$P_0 H \cdot P_1 P_2 = 2 \triangle_3 \sin \gamma = 2 \sqrt{(\triangle_3 + \triangle_3 \cos \gamma)(\triangle_3 - \triangle_3 \cos \gamma)}. \quad (9)$$

From (7–9), together with $P_0 H = \frac{3V}{r_0}$ and $\frac{\triangle_0}{r_0} = \frac{1}{2}(P_1 P_2 + P_2 P_3 + P_3 P_1)$, we have

$$\frac{3V}{r_0} = \sqrt{(\triangle_1 + \triangle_1 \cos \alpha)(\triangle_1 - \triangle_1 \cos \alpha)}$$

$$+ \sqrt{(\triangle_2 + \triangle_2 \cos \beta)(\triangle_2 - \triangle_2 \cos \beta)}$$

$$+ \sqrt{(\triangle_3 + \triangle_3 \cos \gamma)(\triangle_3 - \triangle_3 \cos \gamma)}. \quad (10)$$

Applying Cauchy’s inequality and noting that

$$\triangle_0 = \triangle_1 \cos \alpha + \triangle_2 \cos \beta + \triangle_3 \cos \gamma,$$

we have

$$\left(\frac{3V}{r_0}\right)^2 \leq (\triangle_1 + \triangle_1 \cos \alpha + \triangle_2 + \triangle_2 \cos \beta + \triangle_3 + \triangle_3 \cos \gamma)$$

$$\cdot (\triangle_1 - \triangle_1 \cos \alpha + \triangle_2 - \triangle_2 \cos \beta + \triangle_3 - \triangle_3 \cos \gamma)$$

$$= (\triangle_1 + \triangle_2 + \triangle_3 + \triangle_0)(\triangle_1 + \triangle_2 + \triangle_3 - \triangle_0)$$

$$= (\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2,$$

or

$$(\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2 \geq \left(\frac{3V}{r_0}\right)^2. \quad (11)$$

It is easy to see that equality in (12) holds if and only if

$$\frac{\triangle_1 + \triangle_1 \cos \alpha}{\triangle_1 - \triangle_1 \cos \alpha} = \frac{\triangle_2 + \triangle_2 \cos \beta}{\triangle_2 - \triangle_2 \cos \beta} = \frac{\triangle_3 + \triangle_3 \cos \gamma}{\triangle_3 - \triangle_3 \cos \gamma}.$$  

Equivalently, $\cos \alpha = \cos \beta = \cos \gamma$, or $\alpha = \beta = \gamma$. Similarly, we have

$$(\triangle_2 + \triangle_3 + \triangle_0)^2 - \triangle_1^2 \geq \left(\frac{3V}{r_1}\right)^2, \quad (13)$$

$$(\triangle_3 + \triangle_0 + \triangle_1)^2 - \triangle_2^2 \geq \left(\frac{3V}{r_2}\right)^2, \quad (14)$$

$$(\triangle_0 + \triangle_1 + \triangle_2)^2 - \triangle_3^2 \geq \left(\frac{3V}{r_3}\right)^2. \quad (15)$$
The edge-tangent sphere of a circumscriptible tetrahedron

Summing (12) to (15), we obtain the inequality (6), with equality precisely when all dihedral angles are equal, i.e., when $\mathcal{P}$ is a regular tetrahedron. □

**Remark.** Inequality (6) is obtained by X. Z. Yang in [5].

### 3. Proof of Theorem 3

Since $r = \frac{3V}{\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3}$, it follows from Proposition 6 and Corollary 5 that

$$\ell^2 \geq \frac{6}{\frac{1}{r_0^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2}} \geq \frac{27V^2}{(\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3)^2} = 3r^2.$$  

This completes the proof of Theorem 3.

### 4. A generalization with an open problem

As a generalization of the tetrahedron, we say that an $n$–dimensional simplex is circumscriptible if there is a sphere tangent to each of its edges. The following basic properties of a circumscriptible simplex can be found in [3].

**Theorem 7.** Suppose the edge lengths of an $n$-simplex $\mathcal{P} = P_0P_1\cdots P_n$ are $P_iP_j = a_{ij}$ for $0 \leq i < j \leq n$. The $n$-simplex has an edge-tangent sphere if and only if there exist $x_i$, $i = 0, 1, \ldots, n$, satisfying $a_{ij} = x_i + x_j$ for $0 \leq i \neq j \leq n$. In this case, the radius of the edge-tangent sphere is given by

$$\ell^2 = \frac{D_1}{2D_2},$$

where

$$D_1 = \begin{vmatrix}
-2x_0^2 & 2x_0x_1 & \cdots & 2x_0x_{n-1} \\
2x_0x_1 & -2x_1^2 & \cdots & 2x_1x_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
2x_0x_{n-1} & 2x_1x_{n-1} & \cdots & -2x_{n-1}^2
\end{vmatrix},$$

and

$$D_2 = \begin{vmatrix}
0 & 1 & \cdots & 1 \\
1 & \cdots & 1 \\
\vdots & \vdots & D_1 & \vdots \\
1 & \cdots & 1
\end{vmatrix}.$$  

We conclude this paper with an open problem: for a circumscribable $n$-simplex with a circumscribed sphere of radius $R$, an inscribed sphere of radius $r$ and an edge-tangent sphere of radius $\ell$, prove or disprove that

$$R \geq \sqrt{\frac{2n}{n-1}l} \geq nr.$$
References


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