Ceva Collineations

Clark Kimberling

Abstract. Suppose \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are lines. There exists a unique point \( U \) such that if \( X \in \mathcal{L}_1 \), then \( X^{-1} \odot U \in \mathcal{L}_2 \), where \( X^{-1} \) denotes the isogonal conjugate of \( X \) and \( X^{-1} \odot U \) is the \( X^{-1} \)-Ceva conjugate of \( U \). The mapping \( X \mapsto X^{-1} \odot U \) is the \( U \)-Ceva collineation. It maps every line onto a line and in particular maps \( \mathcal{L}_1 \) onto \( \mathcal{L}_2 \). Examples are given involving the line at infinity, the Euler line, and the Brocard axis. Collineations map cubics to cubics, and images of selected cubics under certain \( U \)-Ceva collineations are briefly considered.

1. Introduction

One of the great geometry books of the twentieth century states [1, p.221] that “Möbius’s invention of homogeneous coordinates was one of the most far-reaching ideas in the history of mathematics”. In triangle geometry, two systems of homogeneous coordinates are in common use: barycentric and trilinear. Trilinears are especially useful when the angle bisectors of a reference triangle \( ABC \) play a central role, as in this note.

Suppose that \( X = x : y : z \) is a point. If at most one of \( x, y, z \) is 0, then the point
\[
X^{-1} = yz : zx : xy
\]
is the isogonal conjugate of \( X \), and if none of \( x, y, z \) is 0, we can write
\[
X^{-1} = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.
\]
A traditional construction for \( X^{-1} \) depends on interior angle bisectors: reflect line \( AX \) in the \( A \)-bisector, \( BX \) in the \( B \)-bisector, \( CX \) in the \( C \)-bisector; then the reflected lines concur in \( X^{-1} \).

The triangle \( A_X B_X C_X \) with vertices
\[
A_X = AX \cap BC, \quad B_X = BX \cap CA, \quad C_X = CX \cap AB
\]
is the cevian triangle of \( X \), and
\[
A_X = 0 : y : z, \quad B_X = x : 0 : z, \quad C_X = x : y : 0.
\]
If \( U = u : v : w \) is a point, then the triangle \( A^U B^U C^U \) with vertices
\[
A^U = -u : v : w, \quad B^U = u : -v : w, \quad C^U = u : v : -w.
\]
is the \textit{anticevian triangle} of $U$. The lines $A_XA_U$, $B_XB_U$, $C_XC_U$ concur in the point

$$u(-uyz + vzx + wxy) : v(uyz - vzx + wxy) : w(uyz + vzx - wyz),$$
called the $X$-Ceva conjugate of $U$ and denoted by $X\odot U$ (see [2, p. 57]). It is easy to verify algebraically that $X\odot(X\odot U) = U$ and that if $P = p : q : r$ is a point, then the equation $P = X\odot U$ is equivalent to

$$X = (ru + pw)(pv + qu) : (pv + qu)(qw + rv) : (qw + rv)(ru + pv)$$

(1)

\quad = \text{cevapoint}(P, U).

A construction of cevapoint $(P, U)$ is given in the Glossary of [3].

One more preliminary will be needed. A \textit{circumconic} is a conic that passes through the vertices, $A, B, C$. Every point $P = p : q : r$, where $pqr \neq 0$, has its own circumconic, given by the equation $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$; indeed, this curve is, loosely speaking, the isogonal conjugate of the line $p\alpha + q\beta + r\gamma = 0$, and the curve is an ellipse, parabola, or hyperbola according as the line meets the circumcircle in 0, 1, or 2 points. The circumcircle is the circumconic having equation $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$.

2. The Mapping $X \mapsto X^{-1}\odot U$

In this section, we present first a lemma: that for given circumconic $\mathcal{P}$ and line $\mathcal{L}$, there is a point $U$ such that the mapping $X \mapsto X\odot U$ takes each point $X$ on $\mathcal{P}$ to a point on $\mathcal{L}$. The lemma easily implies the main theorem of the paper: that the mapping $X \mapsto X^{-1}\odot U$ takes each point of a certain line to $\mathcal{L}$.

\textbf{Lemma 1.} Suppose $L = l : m : n$ and $P = p : q : r$ are points. Let $\mathcal{P}$ denote the circumconic $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$ and $\mathcal{L}$ the line $l\alpha + m\beta + n\gamma = 0$. There exists a unique point $U$ such that if $X \in \mathcal{P}$, then $X\odot U \in \mathcal{L}$. In fact,

$$U = L^{-1}\odot P = p(-lp + mq + nr) : q(lp - mq + nr) : r(lp + mq - nr).$$

\textit{Proof.} We wish to solve the containment $X\odot U \in \mathcal{L}$ for $U$, given that $X \in \mathcal{P}$. That is, we seek $u : v : w$ such that

$$u(-uyz + vzx + wxy)l + v(uyz - vzx + wxy)m + w(uyz + vzx - wyz)n = 0,$$

(2)
given that $X = x : y : z$ is a point satisfying

$$pyz + qzx + qxy = 0.$$  

(3)

Equation (2) is equivalent to

$$u(-ul + vm + wn)yz + v(ul - vm + wn)zx + w(ul + vm - wn)xy = 0,$$

(4)

so that, treating $x : y : z$ as a variable point, equations (3) and (4) represent the same circumconic. Consequently,

$$u(-lu + mv + nw)qr = v(lu - mv + nw)rp = w(lu + mv - nw)pg.$$
In order to solve for \( u : v : w \), we assume, as a first of two cases, that \( p \) and \( q \) are not both 0. Then the equation

\[
u(-lu + mv + nw)qr = v(lu - mv + nw)rp
\]
gives

\[
w = \frac{(mv - lu)(pv + qu)}{n(pv - qu)}.
\] (5)

Substituting for \( w \) in

\[
u(-lu + mv + nw)qr - w(lu + mv - nw)pq = 0
\]
gives

\[
\frac{(mpqv - lpqu + nprv - nqr u - lp^2v + mq^2u)(mv - lu)uv}{2nr(pv - qu)^2} = 0,
\]
so that

\[
u = \frac{(mq - lp + nr)pv}{q(lp - mq + nr)}.
\] (6)

Consequently, for given \( v \), we have

\[
u : v : w = \frac{(mq - lp + nr)pv}{q(lp - mq + nr)} : v : \frac{(mv - lu)(pv + qu)}{n(pv - qu)}.
\]

Substituting for \( u \) from (6), canceling \( v \), and simplifying lead to

\[
u : v : w = p(-lp + mq + nr) : q(lp - mq + nr) : r(lp + mq - nr),
\]
so that \( U = L^{-1} \odot P \).

If, as the second case, we have \( p = q = 0 \), then \( r \neq 0 \) because \( p : q : r \) is assumed to be a point. In this case, one can start with

\[
u(-lu + mv + nw)qr = w(lu + mv - nw)pq
\]
and solve for \( v \) (instead of \( w \) as in (5)) and continue as above to obtain \( U = L^{-1} \odot P \).

The method of proof shows that the point \( U \) is unique.

**Theorem 2.** Suppose \( L_1 \) is the line \( l_1\alpha + m_1\beta + n_1\gamma = 0 \) and \( L_2 \) is the line \( l_2\alpha + m_2\beta + n_2\gamma = 0 \). There exists a unique point \( U \) such that if \( X \in L_1 \), then \( X^{-1} \odot U \in L_2 \).

**Proof.** The hypothesis that \( X \in L_1 \) is equivalent to \( X^{-1} \in \mathcal{P} \), the circumconic having equation \( l_1\beta\gamma + m_1\gamma\alpha + n_1\alpha\beta = 0 \). Therefore, the lemma applies to the circumconic \( \mathcal{P} \) and the line \( L_2 \). \( \square \)

We write the mapping \( X \mapsto X^{-1} \odot U \) as \( C_U(X) = X^{-1} \odot U \) and call \( C_U \) the \( U \)-Ceva collineation. That \( C_U \) is indeed a collineation follows as in [4] from the linearity of \( x, y, z \) in the trilinears

\[
C_U(X) = u(-ux + vy + wz) : v(ux - vy + wz) : w(ux + vy - wz).
\]

This collineation is determined by its action on the four points \( A, B, C, U^{-1} \), with respective images \( A^U, B^U, C^U, U \).
Regarding the surjectivity, or onto-ness, of $C_U$, suppose $F$ is a point on $L_2$; then the equation $X^{-1} \circ U = F$ has as solution
\[ X = \text{cevapoint}(F, U)^{-1}. \]

3. Corollaries

Lemma 1 tells how to find $U$ for given $L$ and $P$. Here, we tell how to find $L$ from given $P$ and $U$ and how to find $P$ from given $U$ and $L$.

**Corollary 3.** Given a circumconic $P$ and a point $U$, there exists a line $L$ such that if $X \in P$, then $X \circ U \in L$.

**Proof.** Assuming there is such a $L$, we have the point $U = L^{-1} \circ P$ as Theorem 2, so that $L^{-1} = \text{cevapoint}(U, P)$, and
\[ L = (\text{cevapoint}(U, P))^{-1}, \]
so that $L$ is the line $(wq + vr)\alpha + (ur + wp)\beta + (vp + uq)\gamma = 0$. It is easy to check that if $X \in P$, then $X \circ U \in L$.

**Corollary 4.** Given a line $L$ and a point $U$, there exists a circumconic $P$ such that if $X \in P$, then $X \circ U \in L$.

**Proof.** Assuming there is such a $L$, we have the point $U = L^{-1} \circ P$, and $P = L^{-1} \circ U$, so that $P$ is the circumconic
\[ u(-ul + vm + wn)\beta\gamma + v(ul - vm + wn)\gamma\alpha + w(uq - vm - wn)\alpha\beta = 0. \]
It is easy to check that if $X \in P$, then $X \circ U \in L$.

4. Examples

4.1. Let $L = P = 1 : 1 : 1$, so that $L_1 = L_2$ is the line $\alpha + \beta + \gamma = 1$. We find $U = 1 : 1 : 1$, so that
\[ C_U(X) = -x + y + z : x - y + z : x + y - z. \]
It is easy to check that $C_U(X) = X$ for every $X$ on the line $\alpha + \beta + \gamma = 1$, such as $X_{44}$ and $X_{513}$. On the line $X_1X_2$ we have
\[ C_U(X) = X \text{ for } X \in \{X_1, X_{899}\}, \]
so that $C_U$ maps $X_1X_2$ onto itself; e.g., $C_U(X_2) = X_{43}$, and $C_U(X_{1201}) = X_8$, and $C_U(X_8) = X_{972}$. On $X_1X_6$ we have fixed points $X_1$ and $X_{44}$, so that $C_U$ maps the line $X_1X_{44}$ to itself. Abbreviating $C_U(X_i) = X_j$ as $X_i \mapsto X_j$, we have, among points on $X_1X_{44},$
\[ X_{1100} \mapsto X_{37} \mapsto X_6 \mapsto X_9 \mapsto X_{1743}. \]
The Euler line, $X_2X_3$, is a link in a chain as indicated by
\[ \cdots \mapsto X_{12}X_{65} \mapsto X_2X_3 \mapsto X_{43}X_{46} \mapsto \cdots \]
4.2. Let \( L = L_1 = X_6 = a : b : c \), so that \( \mathcal{L}_1 \) is the line at infinity and \( \mathcal{P} \) is the circumcircle. Let \( \mathcal{L}_2 \) be the Euler line, given by taking \( L_2 \) in the statement of the theorem to be

\[
X_{647} = a(b^2-c^2)(b^2+c^2-a^2) : b(c^2-a^2)(c^2+a^2-b^2) : c(a^2-b^2)(a^2+b^2-c^2).
\]

The Ceva collineation \( \mathcal{C}_U \) that maps \( \mathcal{L}_1 \) onto \( \mathcal{L}_2 \) is given by

\[
U = X_{523} = a(b^2-c^2) : b(c^2-a^2) : c(a^2-b^2) = \sin(B-C) : \sin(C-A) : \sin(A-B),
\]

and we find

\[
X_{512} \mapsto X_2, \quad X_{520} \mapsto X_4, \quad X_{523} \mapsto X_5, \quad X_{526} \mapsto X_30, \quad X_{2574} \mapsto X_{1312}, \quad X_{2575} \mapsto X_{1313}.
\]

The penultimate of these, namely \( X_{2574} \mapsto X_{1312} \), is of particular interest, as \( X_{2574} = X_{1113} \), where \( X_{1113} \) is a point of intersection of the Euler line and the circumcircle and \( X_{1312} \) is a point of intersection of the Euler line and the nine-point circle; and similarly for \( X_{2575} \mapsto X_{1313} \). The mapping \( \mathcal{C}_U \) carries the Brocard axis, \( X_3X_6 \) onto the line \( X_{115}X_{125} \), where \( X_{115} \) and \( X_{125} \) are the centers of the Kiepert and Jerabek hyperbolas, respectively.

4.3. Let \( L_1 = X_{523} \), so that \( \mathcal{L}_1 \) is the Brocard axis, \( X_3X_6 \), and let \( \mathcal{L}_2 \) be the Euler line, \( X_2X_3 \). Then \( U = X_6 = a : b : c \). The mapping of \( \mathcal{L}_1 \) to \( \mathcal{L}_2 \) is a link in a chain:

\[
\cdots \mapsto X_2X_{39} \mapsto X_2X_6 \mapsto X_3X_6 \mapsto X_2X_3 \mapsto X_6X_{25} \mapsto X_3X_{66} \mapsto \cdots
\]

4.4. Here, we reverse the roles played by the Brocard axis and Euler line in Example 3: let \( \mathcal{L}_1 \) be the Euler line and \( \mathcal{L}_2 \) be the Brocard axis. Then \( U = X_{184} = a^2 \cos A : b^2 \cos B : c^2 \cos C \). A few images of the \( X_{184} \)-Ceva collineation are given here:

\[
X_2 \mapsto X_{32}, \quad X_3 \mapsto X_{571}, \quad X_4 \mapsto X_{577}, \quad X_5 \mapsto X_6, \quad X_{30} \mapsto X_{50}, \quad X_{427} \mapsto X_3.
\]

4.5. Let \( \mathcal{L}_1 = \mathcal{L}_2 = \) Brocard axis. Here,

\[
U = X_5 = \cos(B-C) : \cos(C-A) : \cos(A-B),
\]

the center of the nine-point circle, and

\[
X_{389} \mapsto X_3 \mapsto X_{52} \quad \text{and} \quad X_{570} \mapsto X_6 \mapsto X_{216}.
\]

4.6. Let \( \mathcal{L}_1 = \mathcal{L}_2 = \) the line at infinity, \( X_{30}X_{511} \). Here,

\[
U = X_3 = \cos A : \cos B : \cos C,
\]

the circumcenter. Among line-to-line images under \( X_3 \)-collineation are these:

\[
X_4X_{51} \mapsto \text{Euler line} \mapsto X_3X_{49}, \quad X_6X_{64} \mapsto X_4X_6 \mapsto \text{Brocard axis} \mapsto X_6X_{155}.
\]
5. Cubics

Collineations map cubics to cubics (e.g. [4, p. 23]). In particular, a $U$-Ceva collineation maps a cubic $\Lambda$ that passes through the vertices $A, B, C$ to a cubic $C_U(\Lambda)$ that passes through the vertices $A^U, B^U, C^U$ of the anticevian triangle of $U$.

5.1. Let $U = X_1$, as in §4.1, and let $\Lambda$ be the Thompson cubic, $Z(X_2, X_1)$, with equation

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0.$$ 

Then $C_U(\Lambda)$ circumscribes the excentral triangle, and for selected $X_i$ on $\Lambda$, the image $C_U(X_i)$ is as shown here:

<table>
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<tr>
<th>$X_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>9</th>
<th>57</th>
<th>223</th>
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<td>46</td>
<td>1745</td>
<td>9</td>
<td>1743</td>
<td>165</td>
<td>1750</td>
</tr>
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</table>

5.2. Let $U = X_1$, and let $\Lambda$ be the cubic $Z(X_1, X_75)$, with equation

$$\alpha(\beta^2 - b^2\gamma^2) + \beta(a^2\gamma^2 - c^2\alpha^2) + \gamma(b^2\alpha^2 - a^2\beta^2) = 0.$$ 

For selected $X_i$ on $\Lambda$, the image $C_U(X_i)$ is as shown here:

<table>
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<tr>
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<th>19</th>
<th>31</th>
<th>48</th>
<th>55</th>
<th>56</th>
<th>204</th>
<th>221</th>
</tr>
</thead>
<tbody>
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<td>19</td>
<td>57</td>
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<td>2184</td>
<td>84</td>
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</table>

5.3. Let $U = X_6$, as in §4.3, and let $\Lambda$ be the Thompson cubic. Then $C_U(\Lambda)$ circumscribes the tangential triangle, and for selected $X_i$ on $\Lambda$, the image $C_U(X_i)$ is as shown here:

<table>
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<tr>
<th>$X_i$</th>
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<th>2</th>
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<tbody>
<tr>
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References


Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu