Ten Concurrent Euler Lines

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Dedicated to Svetlozar Doichev

Abstract. Let $F_1$ and $F_2$ denote the Fermat-Toricelli points of a given triangle $ABC$. We prove that the Euler lines of the 10 triangles with vertices chosen from $A, B, C, F_1, F_2$ (three at a time) are concurrent at the centroid of triangle $ABC$.

Given a (positively oriented) triangle $ABC$, construct externally on its sides three equilateral triangles $BCT_a, CAT_b, and ABT_c$ with centers $N_a, N_b, and N_c$ respectively (see Figure 1). As is well known, triangle $N_aN_bN_c$ is equilateral. We call this the first Napoleon triangle of $ABC$.

The same construction performed internally gives equilateral triangles $BCT'_a, CAT'_b$ and $ABT'_c$ with centers $N'_a, N'_b, and N'_c$ respectively, leading to the second Napoleon triangle $N'_aN'_bN'_c$. The centers of both Napoleon triangles coincide with the centroid $M$ of triangle $ABC$.

The lines $AT_a, BT_b$ and $CT_c$ make equal pairwise angles, and meet together with the circumcircles of triangles $BCT_a, CAT_b$, and $ABT_c$ at the first Fermat-Toricelli point $F_1$. Denoting by $\angle XYZ$ the oriented angle $\angle(YX,YZ)$, we have $\angle AF_1B = \angle BF_1C = \angle CF_1A = 120^\circ$. Analogously, the second Fermat-Toricelli point satisfies $\angle AF_2B = \angle BF_2C = \angle CF_2A = 60^\circ$.

Clearly, the sides of the Napoleon triangles are the perpendicular bisectors of the segments joining their respective Fermat-Toricelli points with the vertices of triangle $ABC$ (as these segments are the common chords of the circumcircles of the the equilateral triangles $ABT_c, BCT_a$, etc).

We prove the following interesting theorem.

**Theorem** The Euler lines of the ten triangles with vertices from the set $\{A, B, C, F_1, F_2\}$ are concurrent at the centroid $M$ of triangle $ABC$.

**Proof.** We divide the ten triangles in three types:
(I): Triangle $ABC$ by itself, for which the claim is trivial.
(II): The six triangles each with two vertices from the set $\{A, B, C\}$ and the remaining vertex one of the points $F_1, F_2$.
(III) The three triangles each with vertices $F_1, F_2$, and one from $\{A, B, C\}$.

For type (II), it is enough to consider triangle $ABF_1$. Let $M_c$ be its centroid and $M_C$ be the midpoint of the segment $AB$. Notice also that $N_c$ is the circumcenter of triangle $ABF_1$ (see Figure 2).

![Figure 2](image_url)

Now, the points $C, F_1$ and $T_c$ are collinear, and the points $M, M_c$ and $N_c$ divide the segments $M_C C, M_C F_1$ and $M_C T_c$ in the same ratio 1 : 2. Therefore, they are collinear, and the Euler line of triangle $ABF$ contains $M$. 
For type (III), it is enough to consider triangle $CF_1F_2$. Let $M_c$ and $O_c$ be its centroid and circumcenter. Let also $M_C$ and $M_F$ be the midpoints of $AB$ and $F_1F_2$. Notice that $O_c$ is the intersection of $N_aN_b$ and $N'_aN'_b$ as perpendicular bisectors of $F_1C$ and $F_2C$. Let also $P$ be the intersection of $N_bN_c$ and $N'_bN'_a$, and $F'$ be the reflection of $F_1$ in $M_C$ (see Figure 3).

![Figure 3.](image)

The rotation of center $M$ and angle $120^\circ$ maps the lines $N_aN_b$ and $N'_aN'_b$ into $N_bN_c$ and $N'_aN'_b$ respectively. Therefore, it maps $O_c$ to $P$, and $\angle O_cMP = 120^\circ$. Since $\angle O_cN'_aP = 120^\circ$, the four points $O_c$, $M$, $N'_a$, $P$ are concyclic. The circle containing them also contains $N_b$ since $\angle PN_bO_c = 60^\circ$. Therefore, $\angle O_cMN_b = \angle O_cN'_aN_b$.

The same rotation maps angle $O_cN'_aN_b$ onto angle $PN'_aN_c$, yielding $\angle O_cN'_aN_b = \angle PN'_aN_c$. Since $PN'_a \perp BF_2$ and $N_cN'_c \perp BA$, $\angle PN'_aN_c = \angle F_2BA$.

Since $\angle BF'A = \angle AF_1B = 120^\circ = 180^\circ - \angle AF_2B$, the quadrilateral $AF_2BF'$ is also cyclic and $\angle F_2BA = \angle F_2F'A$. Thus, $\angle F_2F'A = \angle O_cMN_b$.

Now, $AF' \parallel F_1B \perp N_aN_c$ and $N_bM \perp N_aN_c$ yield $AF'' \parallel N_bM$. This, together with $\angle F_2F'A = \angle O_cMN_b$, yields $F''F_2 \parallel MO_c$.

Notice now that the points $M_c$ and $M$ divide the segments $CM_F$ and $CM_C$ in ratio $1 : 2$, therefore $M_cM \parallel CM_F$. The same argument, applied to the segments $F_1F_2$ and $F_1F'$ with ratio $1 : 1$, yields $M_CM_F \parallel F'F_2$.

In conclusion, we obtain $M_cM \parallel F'F_2 \parallel MO_c$. The collinearity of the points $M_c$, $M$ and $O_c$ follows.
References


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