Orthic Quadrilaterals of a Convex Quadrilateral

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Abstract. We introduce the orthic quadrilaterals of a convex quadrilateral, based on the notion of valtitudes. These orthic quadrilaterals have properties analogous to those of the orthic triangle of a triangle.

1. Orthic quadrilaterals

The orthic triangle of a triangle $T$ is the triangle determined by the feet of the altitudes of $T$. The orthic triangle has several and interesting properties (see [2, 4]). In particular, it is the triangle of minimal perimeter inscribed in a given acute-angled triangle (Fagnano’s problem). It is possible to define an analogous notion for quadrilaterals, that is based on the valtitudes of quadrilaterals [6, p.20]. In this case, though, given any quadrilateral we obtain a family of “orthic quadrilaterals”. Precisely, let $A_1A_2A_3A_4$ be a convex quadrilateral, which from now on we will denote by $Q$. We call $v$-parallelogram of $Q$ any parallelogram inscribed in $Q$ and having the sides parallel to the diagonals of $Q$. We denote by $V$ a $v$-parallelogram of $Q$ with vertices $V_i, i = 1, 2, 3, 4$, on the side $A_iA_{i+1}$ (with indices taken modulo 4).

The $v$-parallelograms of $Q$ can be constructed as follows. Fix an arbitrary point $V_1$ on the segment $A_1A_2$. Draw from $V_1$ the parallel to the diagonal $A_1A_3$ and let $V_2$ be the intersection point of this line with the side $A_2A_3$. Draw from $V_2$ the parallel to the diagonal $A_2A_4$ and let $V_3$ be the intersection point of this line with the side $A_3A_4$. Finally, draw from $V_3$ the parallel to the diagonal $A_1A_3$ and let $V_4$ be the intersection point of this line with the side $A_4A_1$. The quadrilateral $V_1V_2V_3V_4$ is a $v$-parallelogram ([6, p.19]). By moving $V_1$ on the segment $A_1A_2$, we obtain all possible $v$-parallelograms of $Q$. The $v$-parallelogram $M_1M_2M_3M_4$, with $M_i$ the midpoint of the segment $A_iA_{i+1}$, is the Varignon’s parallelogram of $Q$.

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Given a \( v \)-parallelogram \( \mathbf{V} \) of \( \mathbf{Q} \), let \( H_i \) be the foot of the perpendicular from \( V_i \) to the line \( A_{i+2}A_{i+3} \). We say that \( H_1H_2H_3H_4 \) is an orthic quadrilateral of \( \mathbf{Q} \), and denote it by \( \mathbf{Q}_o \). Note that \( \mathbf{Q}_o \) may be convex, concave or self-crossing (see Figure 1). The lines \( V_iH_i \) are called the valitudes of \( \mathbf{Q} \) with respect to \( \mathbf{V} \).

The orthic quadrilateral relative to the Varignon’s parallelogram \( (V_i = M_i) \) will be called principal orthic quadrilateral of \( \mathbf{Q} \) and will be denoted by \( \mathbf{Q}_{po} \). The line \( M_iH_i \) is the maltitude of \( \mathbf{Q} \) on the side \( A_{i+2}A_{i+3} \) (see Figure 2).

The study of the orthic quadrilaterals, and in particular of the principal one, allows us to find some properties that are analogous to those of the orthic triangle. In §2 we study the orthic quadrilaterals of an orthogonal quadrilateral, in §3 we consider the case of cyclic and orthodiagonal quadrilaterals. In §4 we find some particular properties of the principal orthic quadrilateral of a cyclic and orthodiagonal quadrilateral. Finally, in §5 we introduce the notion of orthic axis of an orthodiagonal quadrilateral.

2. Orthic quadrilaterals of an orthodiagonal quadrilateral

We recall that the maltitudes of \( \mathbf{Q} \) are concurrent if and only if \( \mathbf{Q} \) is cyclic ([6]). If \( \mathbf{Q} \) is cyclic, the point \( H \) of concurrence of the maltitudes is called anticenter of \( \mathbf{Q} \) (see Figure 3). Moreover, if \( \mathbf{Q} \) is cyclic and orthodiagonal, the anticenter is the common point to the diagonals of \( \mathbf{Q} \) (Brahmagupta’s theorem, [2, p.44]).

In general, if \( \mathbf{Q} \) is cyclic, with circumcenter \( O \) and centroid \( G \), then \( H \) is the symmetric of \( O \) with respect to \( G \), and the line containing the three points \( H, O \) and \( G \) is called Euler line of \( \mathbf{Q} \).

The valitudes of \( \mathbf{Q} \) relative to a \( v \)-parallelogram may concur only if \( \mathbf{Q} \) is cyclic or orthodiagonal [6]. Precisely, when \( \mathbf{Q} \) is cyclic they concur if and only if they are the maltitudes of \( \mathbf{Q} \). When \( \mathbf{Q} \) is orthodiagonal there exists one and only one \( v \)-parallelogram of \( \mathbf{Q} \) with concurrent valitudes. In this case they concur in the
point $D$ common to the diagonals of $Q$, and are perpendicular to the sides of $Q$ through $D$.

**Lemma 1.** If $Q$ is orthodiagonal, the valitudes $V_iH_i$ and $V_{i+1}H_{i+1}$ ($i = 1, 2, 3, 4$) with respect to a $v$-parallelogram $V$ of $Q$ meet on the diagonal $A_{i+1}A_{i+3}$ of $Q$. 

Figure 3.

Figure 4.
Proof. Let $Q$ be orthodiagonal and $V$ a v-parallelogram of $Q$. Let us prove that the altitudes $V_1H_1$ and $V_2H_2$ meet on the line $A_2A_4$ (see Figure 4). The altitudes $V_3K_3$, $V_4K_4$, $A_4H$ of triangle $V_3V_4A_4$ concur at a point $K$ on the line $A_2A_4$. Let $B$ be the common point to $V_1H_1$ and $A_2A_4$. We prove that $B$ is on $V_2H_2$ as well. The quadrilateral $V_1BKV_4$ is a parallelogram, because its opposite sides are parallel. Thus, $BK$ is equal and parallel to $V_1V_4$ and to $V_2V_3$, and the quadrilateral $V_2V_3KB$ is a parallelogram because it has two opposite sides equal and parallel. It follows that $V_2B$ is parallel to $V_3K$, and $B$ lies on $V_2H_2$.

Analogously we can proceed for the other pairs of altitudes. □

Theorem 2. Let $Q$ be orthodiagonal. Let $V$ be a v-parallelogram of $Q$ and $Q_0$ be the orthic quadrilateral of $Q$ relative to $V$. The vertices of $V$ and those of $Q_0$ lie on the same circle.

Proof. In fact, since $Q$ is orthodiagonal, $V$ is a rectangle and it is inscribed in the circle $C$ of diameter $V_1V_3 = V_2V_4$. The vertices of $Q_0$ lie on $C$, because, for example, $\angle V_1H_1V_3$ is a right angle, and $H_1$ lies on $C$ (see Figure 5). □

Note that if $V$ is the Varignon’s parallelogram, the center of the circle $C$ is the centroid $G$ of $Q$. In this case $C$ is known as the eight-point circle of $Q$ (see [1, 3]).

Corollary 3. If $Q$ is orthodiagonal, then each orthic quadrilateral of $Q$, in particular $Q_{po}$, is cyclic.

3. Orthic quadrilaterals of a cyclic and orthodiagonal quadrilateral

The orthic quadrilaterals of $Q$ may not be inscribed in $Q$. In particular, $Q_{po}$ is inscribed in $Q$ if and only if the angles formed by each side of $Q$ with the lines joining its endpoints with the midpoint of the opposite side are acute. It follows that if $Q$ is cyclic and orthodiagonal, then $Q_{po}$ is inscribed in $Q$. 
**Theorem 4.** If \(Q\) is cyclic and orthodiagonal and \(Q_o\) is an orthic quadrilateral of \(Q\) that is inscribed in \(Q\), the valtitudes that detect \(Q_o\) are the internal angle bisectors of \(Q_o\).

![Figure 6.](image)

**Proof.** We prove that the valtitude \(V_1H_1\) is the bisector of \(\angle H_2H_1H_4\) (see Figure 6).

Since \(Q\) is cyclic, we have

\[
\angle A_1A_4A_2 = \angle A_1A_3A_2, \tag{1}
\]

because they are subtended by the same arc \(A_1A_2\). Let \(B\) be the common point to the valtitudes \(V_1H_1\) and \(V_2H_2\) and \(B'\) the common point to the valtitudes \(V_1H_1\) and \(V_4H_4\). The quadrilateral \(BH_1A_4H_2\) is cyclic because the angles in \(H_1\) and in \(H_2\) are right angles; it follows that

\[
\angle H_2H_1B = \angle H_2A_4B, \tag{2}
\]

because they are subtended by the same arc \(H_2B\). Analogously the quadrilateral \(B'H_1A_3H_4\) is cyclic and

\[
\angle B'H_1H_4 = \angle B'A_3H_4. \tag{3}
\]

But, for Lemma 1, \(\angle H_2A_4B = \angle A_1A_4A_2\) and \(\angle B'A_3H_4 = \angle A_1A_3A_2\), then, from (1), (2), (3), it follows that \(\angle H_2H_1V_1 = \angle V_1H_1H_4\).

From Corollary 3 and Theorem 4 applied to the case of maltitudes, we obtain
**Corollary 5.** If $Q$ is cyclic and orthodiagonal, then $Q_{po}$ is bicentric and its centers are the centroid and the anticenter of $Q$ (see Figure 7).

![Figure 7](image)

**Theorem 6.** If $Q$ is cyclic and orthodiagonal, the bimedians of $Q$ are the axes of the diagonals of $Q_{po}$.

![Figure 8](image)
Orthic quadrilaterals of a convex quadrilateral

**Proof.** It is enough to consider the eight-point circle of $Q$ and prove that the bimedian $M_2M_4$ is the axis of the diagonal $H_1H_3$ of $Q_{po}$ (see Figure 8). Note that

$\angle H_3M_4M_2 = \angle H_3H_2M_2$, because they are subtended by the same arc $H_3M_2$. Moreover, $\angle H_3H_2M_2 = \angle H_1H_2M_2$, because $H_2M_2$ bisects $\angle H_1H_2H_3$ (Theorem 4). It follows that $\angle H_3M_4M_2 = \angle H_1H_2M_2$. Then $M_2$ is the midpoint of the arc $H_1H_3$ and $M_2M_4$ is the axis of $H_1H_3$. □

Note that $M_2$ and $M_4$ are the midpoints of the two arcs with endpoints $H_1, H_3$, and $M_1, M_3$ are the midpoints of the two arcs with endpoints $H_2, H_4$.

**Theorem 7.** If $Q$ is cyclic and orthodiagonal, the orthic quadrilaterals of $Q$ inscribed in $Q$ have the same perimeter. Moreover, they have the minimum perimeter of any quadrilateral inscribed in $Q$.

**Proof.** Let $Q$ be cyclic and orthodiagonal and let $Q_o$ be any orthic quadrilateral of $Q$ inscribed in $Q$ (see Figure 9). Let $Q$ be any quadrilateral inscribed in $Q$, different from $Q_o$ . In the figure, $Q_o$ is the red quadrilateral and $Q$ is the blue quadrilateral.

![Figure 9.](image)

Let us consider the reflection in the line $A_1A_4$, that transforms $A_1A_2A_3A_4$ in $A_1B_2B_3A_4$, the reflection in the line $B_3A_4$, that transforms $A_1B_2B_3A_4$ in $B_1C_2B_3A_4$, and the reflection in the line $C_2B_3$, that transforms $B_1C_2B_3A_4$ in...
$C_1C_2B_3B_4$. Let $H$ and $K$ be the vertices of $Q_o$ and $Q$ on the segment $A_1A_2$ respectively, and $H'$ and $K'$ the correspondent points of $H$ and $K$ in the product of the three reflections.

Let us consider the broken line $A_2A_1A_4B_3C_2C_1$. The angles formed by its sides, measured counterclockwise, are $\angle A_1, -\angle A_4, \angle A_3, -\angle A_2$. The sum of these angles is equal to zero, because $Q$ is cyclic, then the final side $C_1C_2$ is parallel to $A_1A_2$. It follows that the segments $HH'$ and $KK'$ are congruent by translation.

For Theorem 4 the valtitudes of $Q$ relative to $Q_o$ are the internal angles bisectors of $Q_o$, then with the three reflections in the lines $A_1A_4, B_3A_4$ and $C_2B_3$, the sides of $Q_o$ will lie on the segment $HH'$, whose length is then equal to the perimeter of $Q_o$. But, the segment $HH'$ is equal to the segment $KK'$, that has the same endpoints of the broken line formed by the sides of $Q$. It follows that the perimeter of $Q$ is greater than or equal to the one of $Q_o$, then the theorem is proved. \hfill $\Box$

4. Properties of the principal orthic quadrilateral of a cyclic and orthodiagonal quadrilateral

Let $Q_o$ be an orthic quadrilateral of $Q$ inscribed in $Q$. Subtracting from $Q$ the quadrilateral $Q_o$, we produce the corner triangles $A_iH_{i+1}H_{i+2}, (i = 1, 2, 3, 4)$.

Lemma 8. Let $Q$ be cyclic and orthodiagonal and let $Q_o$ be an orthic quadrilateral of $Q$ inscribed in $Q$. The triangle $A_iH_{i+1}H_{i+2} (i = 1, 2, 3, 4)$ is similar to the triangle $A_iA_{i+1}A_{i+3}$.

![Figure 10](image-url)
Proof. Let us prove that the triangles $A_1H_2H_3$ and $A_1A_2A_4$ are similar. Then all we need to prove is that $\angle A_1H_2H_3 = \angle A_1A_2A_4$ (see Figure 10).

Let $B$ be the common point to the altitudes $V_1H_1$ and $V_2H_2$. Since the quadrilateral $A_4H_1BH_2$ is cyclic, it is $\angle BH_2H_1 = \angle BA_4H_1$. Moreover, $\angle BH_2H_3 = \angle BH_2H_1$, because the altitude $V_2H_2$ bisects $H_1H_2H_3$. We have $\angle A_3A_1A_2 = \angle A_2A_4A_3$, because $Q$ is cyclic. Then $\angle A_3A_1A_2 = \angle BH_2H_3$. Since $\angle A_1H_2H_3 = 90^\circ - \angle BH_2H_3$ and $\angle A_1A_2A_4 = 90^\circ - \angle A_3A_1A_2$, because $Q$ is orthodiagonal, it is $\angle A_1H_2H_3 = \angle A_1A_2A_4$. □

Suppose now that $Q$ is cyclic and orthodiagonal. Let us find some properties that hold for the principal orthic quadrilateral $Q_{po}$, but not for any orthic quadrilateral of $Q$.

Consider the quadrilateral $Q'$ whose vertices are the points $A'_i$ in which $Q_{po}$ is tangent to its incircle (Corollary 5) and the quadrilateral $Q_t$ whose sides are tangent to the circumcircle of $Q$ at its vertices. We say that $Q_{po}$ is the tangential quadrilateral of $Q'$ and $Q_t$ is the tangential quadrilateral of $Q$.

**Theorem 9.** If $Q$ is cyclic and orthodiagonal, the quadrilaterals $Q'$ and $Q$ and the quadrilaterals $Q_{po}$ and $Q_t$ are correspondent in a homothetic transformation whose center lies on the Euler line of $Q$.

![Figure 11.](image_url)

**Proof.** It suffices to prove that the quadrilaterals $Q'$ and $Q$ are homothetic (see Figure 11).

Let us start proving that the sides of $Q$ are parallel to the sides of $Q'$, for example that $A_1A_4$ is parallel to $A'_1A'_4$. In fact, the altitude $H_1H_2$ is perpendicular to $A_1A_4$; moreover, it bisects $\angle A'_1H_2H_4$, then it is perpendicular to $A'_1A'_4$ also, thus $A_1A_4$ and $A'_1A'_4$ are parallel. It follows, in particular, that the angles of $Q$ are equal to those of $Q'$, precisely $\angle A_i = \angle A'_i$. 
Let us prove now that the sides of \( Q \) are proportional to the sides of \( Q' \). It is \( \angle A_1 H_2 H_3 = \angle H_2 A'_1 A'_4 \), because \( A_1 A_4 \) and \( A'_1 A'_4 \) are parallel, and \( \angle H_2 A'D' = \angle A'B'D' \), because they are subtended by the same arc \( A'D' \), then \( \angle AH_2 H_3 = \angle A'_1 A'_2 A'_4 \). It follows that the triangles \( A_1 H_2 H_3 \) and \( A'_1 A'_2 A'_4 \) are similar. But, for Lemma 8, \( A_1 H_2 H_3 \) is similar to \( A_1 A_2 A_4 \), then the triangles \( A_1 A_2 A_4 \) and \( A'_1 A'_2 A'_4 \) are similar. Analogously it is possible to prove that the triangles \( A_3 A_2 A_4 \) and \( A'_3 A'_2 A'_4 \) are similar. It follows that the sides of \( Q \) are proportional to the sides of \( Q' \). Then it is proved that the quadrilaterals \( Q' \) and \( Q \) are homothetic. Finally, the homothetic transformation that transforms \( Q' \) in \( Q \) transforms the circumcenter \( H \) of \( Q' \) in the circumcenter \( O \) of \( Q \), then the center \( P \) of the homothetic transformation lies on the Euler line of \( Q \). □

It is known that given a circumscriptible quadrilateral and considered the quadrilateral whose vertices are the points of contact of the incircle with the sides, the diagonals of the two quadrilaterals intersect at the same point (see [5] and [7, p.156]). By applying this result to \( Q_t \) and \( Q \) it follows that the diagonals of \( Q_t \) are concurrent in \( H \). Thus the common point to the diagonals of \( Q_{po} \), \( N \), lies on the Euler line. Moreover, \( Q_t \) is cyclic, because \( Q_{po} \) is cyclic, and its circumcenter \( T \) lies on the Euler line (see Figure 12).

![Figure 12.](image-url)
Theorem 10. If $Q$ is cyclic and orthodiagonal and $Q_o$ is an orthic quadrilateral of $Q$ inscribed in $Q$, the perimeter of $Q_o$ is twice the ratio between the area of $Q$ and the radius of the circumcircle of $Q$.

Proof. In fact, from Theorem 7 all orthic quadrilaterals inscribed in $Q$ have the same perimeter, then it suffices to prove the property for $Q_{po}$. The segments $H_1H_2$ and $T_1T_2$ are parallel, because $Q$ and $Q_t$ are homothetic, then they both are perpendicular to $OA_4$, radius of the circumcircle of $Q$ (see Figure 13). It follows that the area of the quadrilateral $OH_1A_4H_2$ is equal to $\frac{1}{2} \cdot OA_4 \cdot H_1H_2$. \qed

Conjecture. If $Q$ is cyclic and orthodiagonal, among all orthic quadrilaterals of $Q$ inscribed in $Q$ the one of maximum area is $Q_{po}$.

The conjecture, which we have been unable to prove, arises from several proofs that we made by using Cabri Géomètre.

5. Orthic axis of an orthodiagonal quadrilateral

Suppose that $Q$ is not a parallelogram. If $Q$ does not have parallel sides, let $R$ be the straight line joining the common points of the lines containing opposite sides of $Q$; if $Q$ is a trapezium, let $R$ be the line parallel to the basis of $Q$ and passing through the common point of the lines containing the oblique sides of $Q$. Let $Q_o$ be any orthic quadrilateral of $Q$ and let $S_i (i = 1, 2, 3, 4)$ be the common point of the lines $H_iH_{i+1}$ and $V_iV_{i+1}$, when these lines intersect (see Figure 14).

Theorem 11. If $Q$ is orthodiagonal and is not a square, for any orthic quadrilateral $Q_o$ of $Q$ the points $S_1, S_2, S_3, S_4$ lie on a line $R$. 

Figure 13.
Proof. Set up an orthogonal coordinate system whose axes are the diagonals of $Q$; then the vertices of $Q$ have coordinates $A_1 = (a_1, 0)$, $A_2 = (0, a_2)$, $A_3 = (a_3, 0)$, $A_4 = (0, a_4)$. The equation of line $R$ is

$$a_2a_4(a_1 + a_3)x + a_1a_3(a_2 + a_4)y - 2a_1a_2a_3a_4 = 0. \tag{4}$$

If $V$ is a v-parallelogram of $Q$, with $x$-coordinate $\alpha$ for the vertex $V_1$, then

$$S_1 = \left( \frac{a_3(a_2(\alpha - a_1) + a_4(\alpha + a_1))}{a_4(a_1 + a_3)}, \frac{a_2(a_1 - \alpha)}{a_1} \right),$$
$$S_2 = \left( \frac{\alpha a_3}{a_1}, \frac{a_2a_4(2a_1^2 - \alpha(a_1 + a_3))}{a_1^2(a_2 + a_4)} \right),$$
$$S_3 = \left( \frac{a_3(a_2(\alpha + a_1) + a_4(\alpha - a_1))}{a_2(a_1 + a_3)}, \frac{a_4(a_1 - \alpha)}{a_1} \right),$$
$$S_4 = \left( \alpha, \frac{a_2a_4(2a_1a_3 - \alpha(a_1 + a_3))}{a_1a_3(a_2 + a_4)} \right).$$

It is not hard to verify that the coordinates of the points $S_i$ satisfy (4). □

We call the line $R$ the orthic axis of $Q$. It is possible to verify that if $Q$ is cyclic and orthodiagonal, i.e., $a_1a_3 = a_2a_4$, the orthic axis of $Q$ is perpendicular to the Euler line of $Q$ (see Figure 15). Moreover, it is known that in a cyclic quadrilateral $Q$ without parallel sides the tangent lines to the circumcircle of $Q$ in two opposite vertices meet on the line joining the common points of the lines containing the opposite sides of $Q$ (see [2, p. 76]). It follows that if $Q$ is cyclic and orthodiagonal and it has not parallel sides, the common points to the lines tangent to the circumcircle of $Q$ in the opposite vertices of $Q$ lie on the orthic axis of $Q$. 

![Figure 14](image-url)
Orthic quadrilaterals of a convex quadrilateral

![Diagram of orthic quadrilaterals](image)

**Figure 15.**

**References**


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