A Note on the Hervey Point of a Complete Quadrilateral

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Abstract. Using the extension of the Sylvester relation to four concyclic points we define the Hervey point of a complete quadrilateral and show that it is the center of a circle congruent to the Miquel circle and that it is the point of concurrence of eight remarkable lines of the complete quadrilateral. We also show that the Hervey point and the Morley point coincide for a particular type of complete quadrilaterals.

1. Introduction

In the present paper we report on several properties of a remarkable point of a complete quadrilateral, the so-called Hervey point. This originated from a problem proposed and solved by F. R. J. Hervey [2] on the concurrence of the four lines drawn through the centers of the nine point circles perpendicular to the Euler lines of the four associated triangles. Here, we use a different approach which seems more efficient than the usual one. We first define the Hervey point by the means of a Sylvester type relation, and, only after that, we look for its properties. Following this way, several interesting results are easily obtained: the Hervey point is the center of a circle congruent to the Miquel circle, and it is the point of concurrence of eight remarkable lines of the complete quadrilateral, obviously including the four lines quoted above.

2. Some properties of the complete quadrilateral and notations

Before we present our approach to the Hervey point, we recall some properties of the complete quadrilateral. Most of them were given by J. Steiner in 1827 [6]. An analysis of the Steiner’s note was recently published by J.-P. Ehrmann [1]. In what follows we quote the theorems as they were labelled by Steiner himself and reported in the Ehrmann’s review. They are referred to as theorems (S-n). The three theorems used in the course of the present paper are:

**Theorem S-1**: The four lines of a complete quadrilateral form four associated triangles whose circumcircles pass through the same point (Miquel point M).

**Theorem S-2**: The centers of the four circumcircles and the Miquel point M lie on the same circle (Miquel circle).

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1Erroneously rendered “the Harvey point” in [5].
**Theorem S-4:** The orthocenters of the four associated triangles lie on the same line (orthocenter or Miquel-Steiner line).

In what follows the indices run from 1 to 4.

The complete quadrilateral is formed with the lines $d_n$. The associated triangle $T_n$ is formed with three lines other than $d_n$, and has circumcenter $O_n$ and orthocenter is $H_n$. The four circumcenters are all distinct, otherwise the complete quadrilateral would be degenerate. According to Theorem S-2, the four centers $O_n$ are concyclic on the Miquel circle with center $O$.

The triangle $\Theta_n$ is formed with the three circumcenters other than $O_n$. It has circumcenter $O$ and orthocenter is $h_n$.

The Euler line $O_nH_n$ cuts the Miquel circle at $O_n$ and another point $N_n$. The triangle $MO_nN_n$ is inscribed in the Miquel circle.

It is worthwhile to mention two more properties:

1. The triangles $\Theta_n$ and $T_n$ are directly similar [1];
2. Each orthocenter $h_n$ is on the corresponding line $d_n$. 
3. The Hervey point of a complete quadrilateral

In order to introduce the Hervey point, we proceed in three steps, successively defining the orthocenter of a triangle, the pseudo-orthocenter of an inscriptable quadrangle and the Hervey point of a complete quadrilateral.

We define the orthocenter $H$ of triangle $ABC$ by the means of the standard Sylvester relation [3, §416]: the vector joining the circumcenter $O$ to the orthocenter $H$ is equal to the sum of the three vectors joining $O$ to the three vertices, namely,

$$\text{OH} = \text{OA} + \text{OB} + \text{OC}.$$ 

It is easy to see that the resultant $AH$ of the two vectors $OB$ and $OC$ is perpendicular to the side $BC$. Therefore the line $AH$ is the altitude drawn through the vertex $A$ and, more generally, the point $H$ is the point of concurrence of the three altitudes. The usual property of the orthocenter is recovered.

We define the pseudo-orthocenter $H$ of the inscriptable quadrangle with the following extension of the Sylvester relation

$$\text{OH} = \text{OP}_1 + \text{OP}_2 + \text{OP}_3 + \text{OP}_4,$$

where $P_1, P_2, P_3, P_4$ are four distinct points lying on a circle with center $O$. We do not go further in the study of this point since many of its properties are similar to those of the Hervey point.

We define the Hervey point $h$ of a complete quadrilateral as the pseudo-orthocenter of the inscriptable quadrangle made up with the four concyclic circumcenters $O_1, O_2, O_3, O_4$ of the triangles $T_1, T_2, T_3, T_4$, which lie on the Miquel circle with center $O$. The extended Sylvester relation reads

$$\text{Oh} = \text{OO}_1 + \text{OO}_2 + \text{OO}_3 + \text{OO}_4.$$

The standard Sylvester relation used for the triangle $\Theta_n$ reads

$$\text{Oh}_n = \sum_{i \neq n} \text{OO}_i.$$ 

This leads to

$$\text{Oh} = \text{Oh}_n + \text{OO}_n$$

for each $n = 1, 2, 3, 4$. From these, we deduce

$$\text{Oh}_n = \text{Oh}_n, \quad h_n h = \text{OO}_n$$

and

$$h_n h_m = O_m O_n \quad \text{for } m \neq n. \quad (1)$$

The last relation implies that the four orthocenters $h_n$ are distinct since the centers $O_n$ are distinct. Figure 2 shows the construction of the Hervey point starting with the orthocenter $h_4$. 
4. Three theorems on the Hervey point

**Theorem 1.** The Hervey point \( h \) is the center of the circle which bears the four orthocenters \( h_n \) of the triangles \( \Theta_n \).

*Proof.* From the relation (1) we deduce that the quadrilateral \( O_m O_n h_m h_n \) is a parallelogram, and its two diagonals \( O_m h_m \) and \( O_n h_n \) intersect at their common midpoints. The four triangles \( \Theta_n \) are endowed with the same role. Therefore the four midpoints of \( O_i h_i \) coincide in a unique point \( S \). By reflection in \( S \) the four points \( O_n \) are transformed into the four orthocenters \( h_n \). Then the orthocenter circle \( (h_1 h_2 h_3 h_4) \) is congruent to the Miquel circle \( (O_1 O_2 O_3 O_4) \) and its center \( \Omega \) corresponds to the center \( O \).

The Hervey point \( h \) and the center \( \Omega \) coincide since, on the one hand \( S \) is the midpoint of \( OO_n \), and on the other hand

\[
Oh = Oh_n + OO_n = OS + Sh_n + OS + SO_n = 2OS.
\]

This is shown in Figure 2. \(\square\)
A part of the theorem (the congruence of the two circles) was already contained in a theorem proved by Lemoine [3, §§265, 417].

**Theorem 2.** The Hervey point $h$ is the point of concurrence of the four altitudes of the triangles $MO_nN_n$ drawn through the vertices $O_n$.

**Proof.** Since the triangles $\Theta_n$ and $T_n$ are directly similar, the circumcenter $O$ and the orthocenter $h_n$ respectively correspond to the circumcenter $O_n$ and orthocenter $H_n$. We have the following equality of directed angles

$$(MO, MO_n) = (Oh_n, O_nH_n) \pmod{\pi}.$$  

On the one hand, the triangle $MOO_n$ is isosceles because the sides $OM$ and $OO_n$ are radii of the Miquel circle. Therefore we have

$$(MO, MO_n) = (O_nM, O_nO) \pmod{\pi}.$$  

On the other hand, since the point $N_n$ is on the Euler line $O_nH_n$, we have

$$(Oh_n, O_nH_n) = (Oh_n, O_nN_n) \pmod{\pi}.$$  

Moreover, it is shown above that $Oh_n = O_nH$. Finally we obtain

$$(O_nM, O_nO) = (O_nh, O_nN_n) \pmod{\pi}.$$  

This equality means that the lines $O_nO$ and $O_nh$ are isogonal with respect to the sides $O_nM$ and $O_nN_n$ of triangle $MO_nN_n$. Since the line $O_nh$ passes through the circumcenter, the line $O_nh$ passes through the orthocenter [3, §253]. Therefore, the line $O_nh$ is the altitude of the triangle $MO_nN_n$ drawn through the vertex $O_n$. In other words, the line $O_nh$ is perpendicular to the chord $MN_n$. This is shown in Figure 3 for the triangle $MO_4N_4$. This result is valid for the four lines $O_nh$. Consequently, they concur at the Hervey point $h$. \hfill \square

**Theorem 3.** The Hervey point $h$ is the point of concurrence of the four perpendicular bisectors of the segments $O_nH_n$ of the Euler lines of the triangles $T_n$.

**Proof.** Since the triangles $\Theta_n$ and $T_n$ are directly similar, the circumcenter $O$ and the orthocenter $h_n$ respectively correspond to the circumcenter $O_n$ and orthocenter $H_n$ and we have the following equality between oriented angles of oriented lines

$$(MO, MO_n) = (Oh_n, O_nH_n) \pmod{2\pi}$$

and the following equality between the side ratios

$$\frac{MO_n}{MO} = \frac{O_nH_n}{Oh_n}.$$  

Since $Oh_n = O_nh$, we convert these equalities into

$$(MO, MO_n) = (O_nh, O_nH_n) \pmod{2\pi}$$

and

$$\frac{MO_n}{MO} = \frac{O_nH_n}{O_nh}.$$
These show that triangles $MOO_n$ and $O_n h H_n$ are directly similar. Since the triangle $MOO_n$ is isosceles, the triangle $O_n h H_n$ is isosceles too. Therefore, the perpendicular bisector of the side $O_n H_n$ passes through the third vertex $h$. This is shown in Figure 4 for the triangles $MOO_4$ and $O_4 h H_4$. This result is valid for the perpendicular bisectors of the four segments $O_n H_n$. Consequently they concur at the Hervey point $h$.

5. The Hervey point of a particular type of complete quadrilaterals

Morley [4] has shown that the four lines drawn through the centers $\omega_n$ of the nine-point circles of the associated triangles $T_n$ perpendicular to the corresponding lines $d_n$, concur at a point $m$ (the Morley point of the complete quadrilateral) on the Miquel-Steiner line.

This result is valid for any complete quadrilateral. This is in contrast with the following theorem which is valid only for a particular type of complete quadrilaterals. Zeeman [7] has shown that if the Euler line of one associated triangle $T_n$ is parallel to the corresponding line $d_n$, then this property is shared by all four associated triangles.

Our Theorem 3 also reads: the four lines drawn through the center $\omega_n$ of the nine point circle perpendicular to the Euler line of the associated triangle concur in the point $h$. 

□
Putting together these three theorems, we conclude that if a complete quadrilateral fulfills the condition of the Zeeman theorem, then the four pairs of lines $\omega_n h$ and $\omega_n m$ coincide. Consequently, their respective points of concurrence $h$ and $m$ coincide too. In other words, the center of the orthocenter circle $(h_1 h_2 h_3 h_4)$ lies on the orthocenter line $H_1 H_2 H_3 H_4$.

**Theorem 4.** *If the Euler line of one associated triangle $T_n$ is parallel to the corresponding line $d_n$, then the Hervey point and the Morley point coincide.*

**References**


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