Synthetic Proofs of Two Theorems Related to the Feuerbach Point

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Abstract. We give synthetic proofs of two theorems on the Feuerbach point of a triangle, one of Paul Yiu, and another of Lev Emelyanov and Tatiana Emelyanova theorem.

1. Introduction

If \( S \) is a point belonging to the circumcircle of triangle \( ABC \), then the images of \( S \) through the reflections with axes \( BC, CA \) and \( AB \) respectively lie on the same line that passes through the orthocenter of \( ABC \). This line is called the Steiner line of \( S \) with respect to triangle \( ABC \).

If a line \( L \) passes through the orthocenter of \( ABC \), then the images of \( L \) through the reflections with axes \( BC, CA \) and \( AB \) are concurrent at one point on the circumcircle of \( ABC \). This point is named the anti-Steiner point of \( L \) with respect to \( ABC \). Of course, \( L \) is Steiner line of \( S \) with respect to \( ABC \) if and only if \( S \) is the anti-Steiner point of \( L \) with respect to \( ABC \). In 2005, using homogenous barycentric coordinates, Paul Yiu [5] established an interesting theorem related to the Feuerbach point of a triangle; see also [3, Theorem 5].

**Theorem 1.** The Feuerbach point of triangle \( ABC \) is the anti-Steiner point of the Euler line of the intouch triangle of \( ABC \) with respect to the same triangle.\(^1\)

In 2009, J. Vonk [4] introduced a geometrically synthetic proof of Theorem 1. In 2001, by calculation, Lev Emelyanov and Tatiana Emelyanova [1] established a theorem that is also very interesting and also related to the Feuerbach point of a triangle.

**Theorem 2.** The circle through the feet of the internal bisectors of triangle \( ABC \) passes through the Feuerbach point of the triangle.

In this article, we present a synthetic proof of Theorem 1, which is different from Vonk’s proof, and one for Theorem 2. We use \((O), I(r), (XYZ)\) to denote respectively the circle with center \( O \), the circle with center \( I \) and radius \( r \), and the circumcircle of triangle \( XYZ \). As in [2, p.12], the directed angle from the line

\(^1\)The anti-Steiner point of the Euler line is called the Euler reflection point in [3].
a to the line b denoted by \((a, b)\). It measures the angle through which a must be rotated in the positive direction in order to become parallel to, or to coincide with, b. Therefore,

(i) \(-90^\circ \leq (a, b) \leq 90^\circ\),

(ii) \((a, b) = (a, c) + (c, b)\),

(iii) If \(a'\) and \(b'\) are the images of a and b respectively under a reflection, then \((a, b) = (b', a')\),

(iv) Four noncollinear points \(A, B, C, D\) are concyclic if and only if \((AC, AD) = (BC, BD)\).

2. Preliminary results

**Lemma 3.** Let \(ABC\) be a triangle inscribed in a circle \((O)\), and \(\mathcal{L}\) an arbitrary line. Let the parallels of \(\mathcal{L}\) through A, B, C intersect the circle at D, E, F respectively. The lines \(\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c\) are the perpendiculars to \(BC, CA, AB\) through \(D, E, F\) respectively.

(a) The lines \(\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c\) are concurrent at a point \(S\) on the circle \((O)\).

(b) The Steiner line of \(S\) with respect to \(ABC\) is parallel to \(\mathcal{L}\).

\[ \begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure} \]

*Proof.* Let \(S\) be the intersection of \(\mathcal{L}_a\) and \((O)\). Let \(\ell\) be the line through \(O\) perpendicular to \(\mathcal{L}\) (see Figure 1).

(a) Because \(A, B,\) and \(C\) are the images of \(D, E,\) and \(F\) through the reflections with axis \(\mathcal{L}\) respectively,

\[
(FE, FD) = (CA, CB).
\]
Therefore, we have
\[(SE, AC) = (SE, SD) + (SD, BC) + (BC, AC)\]
\[= (FE, FD) + 90^\circ + (BC, AC) \quad (F \in (SDE), SD \perp BC)\]
\[= (CA, CB) + 90^\circ + (BC, AC)\]
\[= 90^\circ.\]

Therefore, \(SE\) coincides \(L_b\), i.e., \(S\) lies on \(L_b\). Similarly, \(S\) also lies on \(L_c\), and the three lines \(L_a, L_b, L_c\) are concurrent at \(S\) on the circle \((O)\).

(b) Let \(B_1, C_1\) respectively be the images of \(S\) through the reflections with axes \(CA, AB\). Let \(B_2, C_2\) respectively be the intersection points of \(SB_1, SC_1\) with \(AC, AB\) (see Figure 2). Obviously, \(B_2, C_2\) are the midpoints of \(SB_1, SC_1\) respectively. Thus,
\[B_2C_2 \parallel B_1C_1. \quad (2)\]

Since \(SB_2, SC_2\) are respectively perpendicular to \(AC, AB\),
\[S \in (AB_2C_2). \quad (3)\]

Therefore, we have
\[(B_1C_1, L) = (B_1C_1, AD) \quad (L \parallel AD)\]
\[= (B_2C_2, AD) \quad \text{(by (2))}\]
\[= (B_2C_2, AC_2) + (AB, AD) \quad (B \in AC_2)\]
\[= (B_2S, AS) + (AB, AD) \quad \text{(by (3))}\]
\[= (ES, AS) + (AB, AD) \quad (E \in B_2S)\]
\[= (ED, AD) + (DA, DE) \quad (D \in (SEA))\]
\[= 0^\circ.\]
Therefore, $B_2C_1//L$. This means that the Steiner line of $S$ with respect to triangle $ABC$ is parallel to $L$. \ 

Before we go on to Lemma 4, we review a very interesting concept in plane geometry called the orthopole. Let triangle $ABC$ and the line $L$. $A'$, $B'$, $C'$ are the feet of the perpendiculars from $A$, $B$, $C$ to $L$ respectively. The lines $L_a$, $L_b$, $L_c$ pass through $A'$, $B'$, $C'$ and are perpendicular to $BC$, $CA$, $AB$ respectively. Then $L_a$, $L_b$, $L_c$ are concurrent at one point called the orthopole of the line $L$ with respect to triangle $ABC$. The following result is one of the most important results related to the concept of the orthopole. This result is often attributed to Griffiths, whose proof can be found in [2, pp.246–247].

**Lemma 4.** Let $ABC$ be a triangle inscribed in the circle $(O)$, and $P$ be an arbitrary point other than $O$. The orthopole of the line $OP$ with respect to triangle $ABC$ belongs to the circumcircle of the pedal triangle of $P$ with respect to $ABC$.

**Lemma 5.** Let $ABC$ be a triangle inscribed in $(O)$, $A_1$, $B_1$, $C_1$ are the images of $A$, $B$, $C$ respectively through the symmetry with center $O$. $A_2$, $B_2$, $C_2$ are the images of $O$ through the reflections with axes $BC$, $CA$, $AB$ respectively. $A_3$, $B_3$, $C_3$ are the feet of the perpendiculars from $A$, $B$, $C$ to the lines $OA_2$, $OB_2$, $OC_2$ respectively. Then,

(a) The circles $(OA_1A_2)$, $(OB_1B_2)$, $(OC_1C_2)$ all pass through the anti-Steiner point of the Euler line of triangle $ABC$ with respect to the same triangle.

(b) The circle $(A_3B_3C_3)$ also passes through the same anti-Steiner point.

**Proof.** (a) Let $H$ be the orthocenter of $ABC$. Take the points $D$, $S$ belonging to $(O)$ such that $AD//OH$ and $DS \perp BC$ (see Figure 3).

According to Lemma 3, the Steiner line of $S$ with respect to $ABC$ is parallel to $AD$. On the other hand, the Steiner line of $S$ with respect to $ABC$ passes through $H$. Hence, $OH$ is the Steiner line of $S$ with respect to $ABC$. In other words, $S$ is the anti-Steiner point of the Euler line of $ABC$ with respect to the same triangle.

Let $S_a$ be the intersection of $SD$ and $OH$. By (4), $S_a$ is the images of $S$ through the reflection with axis $BC$. From this, note that $A_2$ is the image of $O$ through the reflection with axis $BC$, we have:

\[ OA_2SS_a \text{ is an isosceles trapezium with } OA_2//S_a. \]  

(5)

Therefore, we have

\[
(A_2O, A_2S) = (S_aO, S_aS) \quad \text{(by (5))}
\]

\[
= (DA, DS) \quad (DA//S_aO \text{ and } D \in S_aS)
\]

\[
= (A_1A, A_1S) \quad (A_1 \in (DAS))
\]

\[
= (A_1O, A_1S) \quad (O \in A_1A).
\]

It follows that $S \in (OA_1A_2)$. Similarly, $S \in (OB_1B_2)$ and $S \in (OC_1C_2)$.

Therefore,

the circles $(OA_1A_2)$, $(OB_1B_2)$, $(OC_1C_2)$ all pass through $S$.  

(6)
From (4) and (6), we can deduce that $(OA_1A_2)$, $(OB_1B_2)$, $(OC_1C_2)$ all pass through the anti-Steiner point of the Euler line of triangle $ABC$ with respect to $ABC$.

(b) Take the points $A_0$, $B_0$, $C_0$ such that $A$, $B$, $C$ are the midpoints of $B_0C_0$, $C_0A_0$, $A_0B_0$ respectively. Let $M$ be the mid-point of $BC$ (see Figure 4). Since $AB//CA_0$ and $AC//BA_0$, $ABA_0C$ is a parallelogram. On the other hand, noting that $HB \perp AC$ and $CA_1 \perp AC$, $HC \perp AB$, and $BA_1 \perp AB$, we have $HB//CA_1$, $HC//BA_1$. This means that $HBA_1C$ is a parallelogram. Thus, $A_0$, $A_1$ are the images of $A$, $H$ respectively through the symmetry with center $M$. Therefore, the vectors $A_1A_0$ and $AH$ are equal.

On the other hand, since $AHS_aD$ is a parallelogram, the vectors $DS_a$ and $AH$ are equal.

Hence, under the translation by the vector $AH$, the points $A_1$, $D$ are transformed into the points $A_0$, $S_a$ respectively. This means that $A_0S_a//A_1D$.

From this, noting that $AD \perp A_1D$ and $AD//OH$, we deduce that

$$A_0S_a \perp OH. \quad (7)$$

On the other hand, because $SS_a \perp BC$ and $BC//B_0C_0$, we have

$$SS_a \perp B_0C_0. \quad (8)$$

From (7) and (8), we see that the orthopole of $OH$ with respect to triangle $A_0B_0C_0$ lies on the line $SS_a$. Similarly, the orthopole of $OH$ with respect to $A_0B_0C_0$ also lies on $SS_b$ and $SS_c$, where $S_b$, $S_c$ are defined in the same way with $S_a$. Thus,

$S$ is the orthopole of $OH$ with respect to triangle $A_0B_0C_0$. \quad (9)

It is also clear that $H$ is the center of the circle $(A_0B_0C_0)$ and

$A_3B_3C_3$ is the pedal triangle of $O$ with respect to triangle $A_0B_0C_0$. \quad (10)
From (9) and (10), and by Lemma 4, we have $S \in (A_3B_3C_3)$. □

**Lemma 6.** If any of the three points in $A$, $B$, $C$, $D$ are not collinear, then the nine-point circles of triangles $BCD$, $CDA$, $DAB$, $ABC$ all pass through one point.

Lemma 6 is familiar and its simple proof can be found in [2, p.242].

3. Main results

3.1. A synthetic proof of Theorem 1. Assume that the circle $I(r)$ inscribed in $ABC$ touches $BC$, $CA$, $AB$ at $A_0$, $B_0$, $C_0$ respectively. Let $A_1$, $B_1$, $C_1$ be the images of $A_0$, $B_0$, $C_0$ respectively through the symmetry with center $I$. Let $A_2$, $B_2$, $C_2$ be the images of $I$ through the reflections with axes $B_0C_0$, $C_0A_0$, $A_0B_0$ respectively. Let $A_3$, $B_3$, $C_3$ be the mid-points of $AI$, $BI$, $CI$ respectively (see Figure 5).

Under the inversion in $I(r)$, the points $A_2$, $B_2$, $C_2$ are transformed into the points $A_3$, $B_3$, $C_3$ respectively. As a result, the circles $(IA_1A_2)$, $(IB_1B_2)$, $(IC_1C_2)$ are transformed into the lines $A_1A_3$, $B_1B_3$, $C_1C_3$ respectively. According to Lemma 5(a), the circles $(IA_1A_2)$, $(IB_1B_2)$, $(IC_1C_2)$ all pass through one point lying on the circle $(I)$, the anti-Steiner point of the Euler line of triangle $A_0B_0C_0$ with respect to the same triangle. We call this point $F$. (11)

Hence, $A_1A_3$, $B_1B_3$, $C_1C_3$ are also concurrent at $F$. (12)
Because $A_1, B_1, C_1$ be the images of $A_0, B_0, C_0$ respectively through the symmetry with center $I$, $A_1B_1, A_1C_1$ are parallel to $A_0B_0, A_0C_0$ respectively.

From this, noting that $A_0B_0, A_0C_0$ are perpendicular to $IC, IB$ respectively, we deduce that

$$A_1B_1, A_1C_1$$

are perpendicular to $IC, IB$.  \hspace{1cm} (13)

Let $M$ be the mid-point of $BC$. Noting that $B_3, C_3$ are the mid-points of $BI, CI$ respectively, we have

$$IC \parallel MB_3 \quad \text{and} \quad IB \parallel MC_3.$$  \hspace{1cm} (14)

Therefore, we have

$$(FB_3, FC_3) = (FB_1, FC_1)$$

(by (12))

$$= (A_1B_1, A_1C_1) \quad (A_1 \in (FB_1C_1))$$

$$= (IC, IB)$$

(by (13))

$$= (MB_3, MC_3)$$

(by (14)).

From this, $F \in (MB_3C_3)$, the nine-point circle of triangle $IBC$.

Similarly, $F$ also belongs to the nine-point circles of triangles $ICA, IAB$.

Thus, from Lemma 6, $F$ belongs to the nine-point circle of triangle $ABC$. This means that

$$F \text{ is the Feuerbach point of triangle } ABC.$$  \hspace{1cm} (15)

From (11) and (15), $F$ is not only the anti-Steiner point of the Euler line of $A_0B_0C_0$ with respect to $A_0B_0C_0$, but also the Feuerbach point of $ABC$.

Thus, we can conclude that the Feuerbach point of $ABC$ is the anti-Steiner point of the Euler line of $A_0B_0C_0$.  

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**Figure 5.**
3.2. A synthetic proof of Theorem 2. Suppose that the inscribed circle $I(\rho)$ of triangle $ABC$ touches $BC, CA, AB$ at $A_0, B_0, C_0$ respectively. Let $A', B', C'$ be the intersections of $AI, BI, CI$ with $BC, CA, AB$ respectively; $A'', B'', C''$ be the feet of the perpendiculards from $A_0, B_0, C_0$ to $AI, BI, CI$ respectively and $F$ be the Feuerbach point of $ABC$ (see Figure 6).

From Lemma 5(b) and Theorem 1, $F \in (A''B''C'')$. (16)

On the other hand, under inversion in the incircle $I(\rho)$, $F, A', B', C'$ are transformed into $F, A'', B'', C''$ respectively. (17)

From (16) and (17), we can conclude that In conclusion, the circumcircle of $A'B'C''$ passes through the Feuerbach point $F$ of $ABC$.

References


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