Similar Metric Characterizations of Tangential and Extangential Quadrilaterals

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Abstract. We prove five necessary and sufficient conditions for a convex quadrilateral to have an excircle and compare them to similar conditions for a quadrilateral to have an incircle.

1. Introduction

There are a lot of more or less well known characterizations of tangential quadrilaterals, that is, convex quadrilaterals with an incircle. This circle is tangent at the inside of the quadrilateral to all four sides. Many of these necessary and sufficient conditions were either proved or reviewed in [8]. In this paper we shall see that there are a few very similar looking characterizations for a convex quadrilateral to have an excircle. This is a circle that is tangent at the outside of the quadrilateral to the extensions of all four sides. Such a quadrilateral is called an extangential quadrilateral in [13, p.44], see Figure 1.

Figure 1. An extangential quadrilateral and its excircle

We start by reviewing and commenting on the known characterizations of extangential quadrilaterals and the similar ones for tangential quadrilaterals. It is well known that a convex quadrilateral is tangential if and only if the four internal angle...
bisectors to the vertex angles are concurrent. Their common point is the incent-
er, that is, the center of the incircle. A convex quadrilateral is extangential if and
only if six angles bisectors are concurrent, which are the internal angle bisectors
at two opposite vertex angles, the external angle bisectors at the other two vertex
angles, and the external angle bisectors at the angles formed where the extensions
of opposite sides intersect. Their common point is the excenter ($E$ in Figure 1).

The most well known and useful characterization of tangential quadrilaterals is
the Pitot theorem, that a convex quadrilateral with sides $a, b, c, d$ has an incircle if
and only if opposite sides have equal sums,

$$a + c = b + d.$$  

For the existence of an excircle, the similar characterization states that the adjacent
sides shall have equal sums. This is possible in two different ways. There can
only be one excircle to a quadrilateral, and the characterization depends on which
pair of opposite vertices the excircle is outside of. It is easy to realize that it must
be outside the vertex (of the two considered) with the biggest angle.\(^3\) A convex
quadrilateral $ABCD$ has an excircle outside one of the vertices $A$ or $C$ if and only if

$$a + b = c + d \quad (1)$$

according to [2] and [10, p.69]. This was proved by the Swiss mathematician Jakob
Steiner (1796–1863) in 1846 (see [3, p.318]). By symmetry ($b \leftrightarrow d$), there is an
excircle outside one of the vertices $B$ or $D$ if and only if

$$a + d = b + c. \quad (2)$$

From (1) and (2), we have that a convex quadrilateral with sides $a, b, c, d$ has an
excircle if and only if

$$|a - c| = |b - d|$$

which resembles the Pitot theorem. There is however one exception to these char-
acterizations. The existence of an excircle is dependent on the fact that the exten-
sions of opposite sides in the quadrilateral intersect, otherwise the circle can never
be tangent to all four extensions. Therefore there is no excircle to either of a trape-
zoid, a parallelogram, a rhombus, a rectangle or a square even though (1) or (2)
is satisfied in many of them, since they have at least one pair of opposite parallel
sides.\(^4\)

In [8, p.66] we reviewed two characterizations of tangential quadrilaterals re-
garding the extensions of the four sides. Let us take another look at them here. If
$ABCD$ is a convex quadrilateral where opposite sides $AB$ and $CD$ intersect at $E$,
and the sides $AD$ and $BC$ intersect at $F$ (see Figure 2), then $ABCD$ is a tangential
quadrilateral if and only if either of the following conditions holds:

$$AE + CF = AF + CE, \quad (3)$$

$$BE + BF = DE + DF. \quad (4)$$

\(^3\)Otherwise the circle can never be tangent to all four extensions.

\(^4\)The last four of these quadrilaterals can be considered to be extangential quadrilaterals with
infinite exradius, see Theorem 8.
The history of these conditions are discussed in [14] together with the corresponding conditions for extangential quadrilaterals. In our notations, $ABCD$ has an excircle outside one of the vertices $A$ or $C$ if and only if either of the following conditions holds:

$$AE + CE = AF + CF,$$

(5)

$$BE + DE = BF + DF.$$  (6)

These conditions were stated somewhat differently in [14] with other notations. Also, there it was not stated that the excircle can be outside $A$ instead of $C$, but that is simply a matter of making the change $A \leftrightarrow C$ in (5) to see that the condition is unchanged. How about an excircle outside of $B$ or $D$? By making the changes $A \leftrightarrow D$ and $B \leftrightarrow C$ (to preserve that $AB$ and $CD$ intersect at $E$) we find that the conditions (5) and (6) are still the same. According to [14], conditions (3) and (5) were proved by Jakob Steiner in 1846. In 1973, Howard Grossman (see [5]) contributed with the two additional conditions (4) and (6).

From a different point of view, (3) and (5) can be considered to be necessary and sufficient conditions for when a concave quadrilateral $AECF$ has an “incircle” (a circle tangent to two adjacent sides and the extensions of the other two) or an excircle respectively. Then (4) and (6) are necessary and sufficient conditions for a complex quadrilateral $BEDF$ to have an excircle.\(^5\)

Another related theorem is due to the Australian mathematician M. L. Urquhart (1902–1966). He considered it to be “the most elementary theorem of Euclidean geometry”. It was originally stated using only four intersecting lines. We restate it in the framework of a convex quadrilateral $ABCD$, where opposite sides intersect at $E$ and $F$, see Figure 2. Urquhart’s theorem states that if $AB + BC = AD + DC$, then $AE + EC = AF + FC$. In 1976 Dan Pedoe wrote about this theorem (see [12]), where he concluded that the proof by purely geometrical methods is not elementary and that he had been trying to find such a proof that did not involve a circle (the excircle to the quadrilateral). Later that year, Dan Sokolowsky took up

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\(^5\)Equations (4) and (6) can then be merged into one as $|BE - DF| = |BF - DE|$.
that challenge and gave an elementary “no-circle” proof in [15]. In 2006, Mowaf-faq Hajja gave a simple trigonometric proof (see [6]) that the two equations in Urquhart’s theorem are equivalent. According to (1) and (5), they are both characterizations of an extangential quadrilateral $ABCD$.

2. Characterizations with subtriangle circumradii

In [9, pp.23–24] we proved that if the diagonals in a convex quadrilateral $ABCD$ intersect at $P$, then it has an incircle if and only if

$$R_1 + R_3 = R_2 + R_4$$

where $R_1$, $R_2$, $R_3$ and $R_4$ are the circumradii in the triangles $ABP$, $BCP$, $CDP$ and $DAP$ respectively, see Figure 3.

![Figure 3. The subtriangle circumcircles](image)

There are the following similar conditions for a quadrilateral to have an excircle.

**Theorem 1.** Let $R_1$, $R_2$, $R_3$, $R_4$ be the circumradii in the triangles $ABP$, $BCP$, $CDP$, $DAP$ respectively in a convex quadrilateral $ABCD$ where the diagonals intersect at $P$. It has an excircle outside one of the vertices $A$ or $C$ if and only if

$$R_1 + R_2 = R_3 + R_4$$

and an excircle outside one of the vertices $B$ or $D$ if and only if

$$R_1 + R_4 = R_2 + R_3.$$ 

**Proof.** According to the extended law of sines, the sides satisfies $a = 2R_1 \sin \theta$, $b = 2R_2 \sin \theta$, $c = 2R_3 \sin \theta$ and $d = 2R_4 \sin \theta$, where $\theta$ is the angle between the diagonals.\(^6\) see Figure 3. Thus

$$a + b - c - d = 2 \sin \theta(R_1 + R_2 - R_3 - R_4)$$

\(^6\)We used that $\sin (\pi - \theta) = \sin \theta$ to get two of the formulas.
and
\[ a + d - b - c = 2 \sin \theta (R_1 + R_4 - R_2 - R_3). \]

From these we directly get that
\[ a + b = c + d \iff R_1 + R_2 = R_3 + R_4 \]
and
\[ a + d = b + c \iff R_1 + R_4 = R_2 + R_3 \]
since \( \sin \theta \neq 0 \). By (1) and (2) the conclusions follow. \( \square \)

3. Characterizations concerning the diagonal parts

In [7] Larry Hoehn made a few calculations with the law of cosines to prove that in a convex quadrilateral \( ABCD \) with sides \( a, b, c, d \),
\[ e f g h (a+c+b+d)(a+c-b-d) = (a g h + c e f + b e h + d f g)(a g h + c e f - b e h - d f g) \]
where \( e, f, g, h \) are the distances from the vertices \( A, B, C, D \) respectively to the diagonal intersection (see Figure 4). Using the Pitot theorem \( a + c = b + d \), we get that the quadrilateral is tangential if and only if
\[ a g h + c e f = b e h + d f g. \] (7)

Now we shall prove that there are similar characterizations for the quadrilateral to have an excircle.

**Theorem 2.** Let \( e, f, g, h \) be the distances from the vertices \( A, B, C, D \) respectively to the diagonal intersection in a convex quadrilateral \( ABCD \) with sides \( a, b, c, d \). It has an excircle outside one of the vertices \( A \) or \( C \) if and only if
\[ a g h + b e h = c e f + d f g \]
and an excircle outside one of the vertices \( B \) or \( D \) if and only if
\[ a g h + d f g = b e h + c e f. \]

**Proof.** In [7] Hoehn proved that in a convex quadrilateral,
\[ e f g h (a^2 + c^2 - b^2 - d^2) = a^2 g^2 h^2 + c^2 e^2 f^2 - b^2 e^2 f^2 - d^2 f^2 g^2. \]

Now adding \( e f g h (-2ac + 2bd) \) to both sides, this is equivalent to
\[ e f g h ((a - c)^2 - (b - d)^2) = (a g h - c e f)^2 - (b e h - d f g)^2 \]
which is factored as
\[ e f g h (a-c+b-d)(a-c-b+d) = (a g h - c e f + b e h + d f g)(a g h - c e f - b e h - d f g). \]
The left hand side is zero if and only if \( a + b = c + d \) or \( a + d = b + c \) and the right hand side is zero if and only if \( a g h + b e h = c e f + d f g \) or \( a g h + d f g = b e h + c e f \).

To show that the first equality from both sides are connected and that the second equality from both sides are also connected, we study a special case. In a kite where \( a = d \) and \( b = c \) and also \( f = h \), the two equalities \( a + b = c + d \) and \( a g h + b e h = c e f + d f g \) are satisfied, but none of the others. This proves that they
are connected. In the same way, using another kite, the other two are connected and we have that

\[ a + b = c + d \iff agh + beh = cef + dfg \]

and

\[ a + d = b + c \iff agh + dfg = beh + cef. \]

This completes the proof according to (1) and (2).

**Remark.** The characterization (7) had been proved at least three different times before Hoehn did it. It appears as part of a proof of an inverse inradii characterization of tangential quadrilaterals in [16] and [17]. It was also proved in [11, Proposition 2 (e)]. All of the four known proofs used different notations.

### 4. Characterizations with subtriangle altitudes

In 2009, Nicușor Minculete gave two different proofs (see [11]) that a convex quadrilateral \(ABCD\) has an incircle if and only if the altitudes \(h_1, h_2, h_3, h_4\) from the diagonal intersection \(P\) to the sides \(AB, BC, CD, DA\) in triangles \(ABP, BCP, CDP, DAP\) respectively satisfy

\[ \frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}. \]  

(8)

This characterization of tangential quadrilaterals had been proved as early as 1995 in Russian by Vasileyev and Senderov [16]. Another Russian proof was given in 2004 by Zaslavsky [18]. To prove that (8) holds in a tangential quadrilateral (i.e. not the converse) was a problem at the 2009 mathematics Olympiad in Germany [1]. All of these but the 1995 proof used other notations.

![Figure 4. The subtriangle altitudes \(h_1, h_2, h_3,\) and \(h_4\)](image)

Here we will give a short fifth proof that (8) is a necessary and sufficient condition for a convex quadrilateral to have an incircle using the characterization (7).
By expressing twice the area of \( ABP, BCP, CDP, DAP \) in two different ways, we have the equalities (see Figure 4)

\[
\begin{align*}
ah_1 &= ef \sin \theta, \\
bh_2 &= fg \sin \theta, \\
ch_3 &= gh \sin \theta, \\
dh_4 &= he \sin \theta
\end{align*}
\]

where \( \theta \) is the angle between the diagonals. \(^7\) Hence

\[
\left( \frac{1}{h_1} + \frac{1}{h_3} - \frac{1}{h_2} - \frac{1}{h_4} \right) \sin \theta = \frac{a}{ef} + \frac{b}{fg} - \frac{c}{gh} = \frac{agh + cef - beh - dfg}{efgh}.
\]

Since \( \sin \theta \neq 0 \), we have that

\[
\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4} \iff agh + cef = beh + dfg
\]

which by (7) proves that (8) is a characterization of tangential quadrilaterals.

Now we prove the similar characterizations of extangential quadrilaterals.

**Theorem 3.** Let \( h_1, h_2, h_3, h_4 \) be the altitudes from the diagonal intersection \( P \) to the sides \( AB, BC, CD, DA \) in the triangles \( ABP, BCP, CDP, DAP \) respectively in a convex quadrilateral \( ABCD \). It has an excircle outside one of the vertices \( A \) or \( C \) if and only if

\[
\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}
\]

and an excircle outside one of the vertices \( B \) or \( D \) if and only if

\[
\frac{1}{h_1} + \frac{1}{h_4} = \frac{1}{h_2} + \frac{1}{h_3}.
\]

**Proof.** The four equations (9) yields

\[
\left( \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{h_3} - \frac{1}{h_4} \right) \sin \theta = \frac{a}{ef} + \frac{b}{fg} - \frac{c}{gh} = \frac{agh + beh - cef - dfg}{efgh}.
\]

Since \( \sin \theta \neq 0 \), we have that

\[
\frac{1}{h_1} + \frac{1}{h_2} = \frac{1}{h_3} + \frac{1}{h_4} \iff agh + beh = cef + dfg
\]

which by Theorem 2 proves the first condition in the theorem. The second is proved in the same way. \( \square \)

\(^7\)Here we have used that \( \sin (\pi - \theta) = \sin \theta \) in two of the equalities.
5. Iosifescu’s characterization for excircles

According to [11, p. 113], Marius Iosifescu proved in 1954 that a convex quadrilateral $ABCD$ has an incircle if and only if

$$\tan \frac{x}{2} \tan \frac{z}{2} = \tan \frac{y}{2} \tan \frac{w}{2}$$

where $x = \angle ABD$, $y = \angle ADB$, $z = \angle BDC$ and $w = \angle DBC$, see Figure 5. That proof was given in Romanian, but an English one was given in [8, pp. 75–77].

![Figure 5. Angles in Iosifescu’s characterization](image)

There are similar characterizations for a quadrilateral to have an excircle, which we shall prove in the next theorem.

**Theorem 4.** Let $x = \angle ABD$, $y = \angle ADB$, $z = \angle BDC$ and $w = \angle DBC$ in a convex quadrilateral $ABCD$. It has an excircle outside one of the vertices $A$ or $C$ if and only if

$$\tan \frac{x}{2} \tan \frac{z}{2} = \tan \frac{y}{2} \tan \frac{w}{2}$$

and an excircle outside one of the vertices $B$ or $D$ if and only if

$$\tan \frac{x}{2} \tan \frac{y}{2} = \tan \frac{z}{2} \tan \frac{w}{2}.$$

**Proof.** In [8], Theorem 7, we proved by using the law of cosines that

$$1 - \cos x = \frac{(d + a - q)(d - a + q)}{2aq}, \quad 1 + \cos x = \frac{(a + q + d)(a + q - d)}{2aq},$$

$$1 - \cos y = \frac{(a + d - q)(a - d + q)}{2dq}, \quad 1 + \cos y = \frac{(d + q + a)(d + q - a)}{2dq},$$

$$1 - \cos z = \frac{(b + c - q)(b - c + q)}{2cq}, \quad 1 + \cos z = \frac{(c + q + b)(c + q - b)}{2cq},$$

$$1 - \cos w = \frac{(c + b - q)(c - b + q)}{2bq}, \quad 1 + \cos w = \frac{(b + q + c)(b + q - c)}{2bq},$$
where \( a = AB, b = BC, c = CD, d = DA \) and \( q = BD \) in quadrilateral \( ABCD \). Using these and the trigonometric identity

\[
\tan^2 \frac{u}{2} = \frac{1 - \cos u}{1 + \cos u},
\]

the second equality in the theorem is equivalent to

\[
\begin{align*}
(d + a - q)^2(d - a + q)(a - d + q)(c + q - b)(b + q - c) \\
\quad = (b + c - q)^2(b - c + q)(c - b + q)(a + q + d)(a + q - d)(d + q - a)
\end{align*}
\]

This is factored as

\[
4qQ_1(a + d - b - c) \left( (a + d)(b + c) - q^2 \right) = 0 \quad (10)
\]

where

\[
Q_1 = \frac{(a - d + q)(d - a + q)(b - c + q)(c - b + q)}{16abcdq^4}
\]

is a positive expression according to the triangle inequality. We also have that \( a + d > q \) and \( b + c > q \), so \( (a + d)(b + c) > q^2 \). Hence we have proved that

\[
\tan \frac{x}{2} \tan \frac{y}{2} = \tan \frac{z}{2} \tan \frac{w}{2} \iff a + d = b + c
\]

which according to (2) shows that the second equality in the theorem is a necessary and sufficient condition for an excircle outside of \( B \) or \( D \).

The same kind of reasoning for the first equality in the theorem yields

\[
4qQ_2(a + b - c - d) \left( (a + b)(c + d) - q^2 \right) = 0 \quad (11)
\]

where \( (a + b)(c + d) > q^2 \) and

\[
Q_2 = \frac{(a - b + q)(b - a + q)(d - c + q)(c - d + q)}{16abcdq^4} > 0.
\]

Hence

\[
\tan \frac{x}{2} \tan \frac{w}{2} = \tan \frac{y}{2} \tan \frac{z}{2} \iff a + b = c + d
\]

which according to (1) shows that the first equality in the theorem is a necessary and sufficient condition for an excircle outside of \( A \) or \( C \).

\[\Box\]

6. Characterizations with escribed circles

All convex quadrilaterals \( ABCD \) have four circles, each of which is tangent to one side and the extensions of the two adjacent sides. In a triangle they are called the excircles, but for quadrilaterals we have reserved that name for a circle tangent to the extensions of all four sides. For this reason we will call a circle tangent to one side of a quadrilateral and the extensions of the two adjacent sides an escribed circle. The four of them have the interesting property that their centers form a cyclic quadrilateral. If \( ABCD \) has an incircle, then its center is also the intersection of the diagonals in that cyclic quadrilateral [4, pp.1–2, 5].

\[8\]In triangle geometry the two names excircle and escribed circle are synonyms.
First we will prove a new characterization for when a convex quadrilateral has an incircle that concerns the escribed circles.

**Theorem 5.** A convex quadrilateral with consecutive escribed circles of radii $R_a$, $R_b$, $R_c$ and $R_d$ is tangential if and only if

$$R_a R_c = R_b R_d.$$

![Figure 6. The four escribed circles](image)

**Proof.** We consider a convex quadrilateral $ABCD$ where the angle bisectors intersect at $I_a$, $I_b$, $I_c$ and $I_d$. Let the distances from these four intersections to the sides of the quadrilateral be $r_a$, $r_b$, $r_c$ and $r_d$, see Figure 6. Then we have

$$r_a \left( \cot \frac{A}{2} + \cot \frac{B}{2} \right) = a = R_a \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right),$$

$$r_b \left( \cot \frac{B}{2} + \cot \frac{C}{2} \right) = b = R_b \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right),$$

$$r_c \left( \cot \frac{C}{2} + \cot \frac{D}{2} \right) = c = R_c \left( \tan \frac{C}{2} + \tan \frac{D}{2} \right),$$

$$r_d \left( \cot \frac{D}{2} + \cot \frac{A}{2} \right) = d = R_d \left( \tan \frac{D}{2} + \tan \frac{A}{2} \right).$$
From two of these we get
\[ r_b r_d \left( \frac{\cot B}{2} + \frac{\cot C}{2} \right) \left( \frac{\cot A}{2} + \frac{\cot D}{2} \right) = R_b R_d \left( \tan B - \tan C \right) \left( \tan A - \tan D \right), \]
whence
\[ r_b r_d \left( \frac{\cos A}{2} \sin D + \sin A \cos D}{2} \right) \left( \frac{\cos B}{2} \sin C + \sin B \cos C}{2} \right) = R_b R_d \left( \frac{\sin B}{2} \cos C + \cos B \sin C}{2} \right) \left( \frac{\sin A}{2} \cos D + \cos A \sin D}{2} \right). \]
This is equivalent to
\[ \frac{r_b r_d}{R_b R_d} = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{D}{2}. \]  
(12)

By symmetry we also have
\[ r_a r_c = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{D}{2}; \]  
(13)
so
\[ \frac{r_a r_c}{R_a R_c} = \frac{r_b r_d}{R_b R_d}. \]  
(14)

The quadrilateral is tangential if and only if the angle bisectors are concurrent, which is equivalent to \( I_a \equiv I_b \equiv I_c \equiv I_d \). This in turn is equivalent to that \( r_a = r_b = r_c = r_d \). Hence by (14) the quadrilateral is tangential if and only if \( R_a R_c = R_b R_d \).

We also have the following formulas. They are not new, and can easily be derived in a different way using only similarity of triangles.

**Corollary 6.** In a bicentric quadrilateral\(^9\) and a tangential trapezoid with consecutive escribed circles of radii \( R_a, R_b, R_c \) and \( R_d \), the incircle has the radius
\[ r = \sqrt{R_a R_c} = \sqrt{R_b R_d}. \]

**Proof.** In these quadrilaterals, \( A + C = \pi = B + D \) or \( A + D = \pi = B + C \) (if we assume that \( AB \parallel DC \)). Thus
\[ \tan \frac{A}{2} \tan \frac{C}{2} = \tan \frac{B}{2} \tan \frac{D}{2} = 1 \]
or
\[ \tan \frac{A}{2} \tan \frac{D}{2} = \tan \frac{B}{2} \tan \frac{C}{2} = 1. \]
In either case the formulas for the inradius follows directly from (13) and (12), since \( r = r_a = r_b = r_c = r_d \) when the quadrilateral has an incircle.

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\(^9\)This is a quadrilateral that has both an incircle and a circumcircle.
In comparison to Theorem 5 we have the following characterizations for an extangential quadrilateral.

**Theorem 7.** Let a convex quadrilateral $ABCD$ have consecutive escribed circles of radii $R_a$, $R_b$, $R_c$ and $R_d$. The quadrilateral has an excircle outside one of the vertices $A$ or $C$ if and only if

$$R_a R_b = R_c R_d$$

and an excircle outside one of the vertices $B$ or $D$ if and only if

$$R_a R_d = R_b R_c.$$

**Proof.** We consider a convex quadrilateral $ABCD$ where two opposite internal and two opposite external angle bisectors intersect at $E_a$, $E_c$, $E_b$ and $E_d$. Let the distances from these four intersections to the sides of the quadrilateral be $\rho_a$, $\rho_c$, $\rho_b$ and $\rho_d$ respectively, see Figure 7. Then we have

$$\rho_a \left( \cot \frac{A}{2} - \tan \frac{B}{2} \right) = a = R_a \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right),$$

$$\rho_b \left( \tan \frac{B}{2} - \cot \frac{C}{2} \right) = b = R_b \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right),$$

$$\rho_c \left( \tan \frac{D}{2} - \cot \frac{C}{2} \right) = c = R_c \left( \tan \frac{C}{2} + \tan \frac{D}{2} \right),$$

$$\rho_d \left( \cot \frac{A}{2} - \tan \frac{D}{2} \right) = d = R_d \left( \tan \frac{D}{2} + \tan \frac{A}{2} \right).$$

![Figure 7. Intersections of four angle bisectors](Image)
Using the first two of these, we get
\[ \rho_a \rho_b \left( \cot \frac{A}{2} - \tan \frac{B}{2} \right) \left( \tan \frac{B}{2} - \cot \frac{C}{2} \right) \]
\[ = R_a R_b \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right), \]
whence
\[ \rho_a \rho_b \left( \frac{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2}}{\sin \frac{A}{2} \cos \frac{B}{2}} \right) \left( \frac{\sin \frac{B}{2} \sin \frac{C}{2} - \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{B}{2} \sin \frac{C}{2}} \right) \]
\[ = R_a R_b \left( \frac{\sin \frac{A}{2} \cos \frac{B}{2} + \cos \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \right) \left( \frac{\sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \right). \]
This is equivalent to
\[ \rho_a \rho_b \frac{\cos \frac{A+B}{2} \cos \frac{B+C}{2}}{\sin \frac{A}{2} \cos^2 \frac{B}{2} \sin \frac{C}{2}} = R_a R_b \frac{\sin \frac{A+B}{2} \sin \frac{B+C}{2}}{\cos \frac{A}{2} \cos^2 \frac{B}{2} \cos \frac{C}{2}}, \]
which in turn is equivalent to
\[ \frac{\rho_a \rho_b}{R_a R_b} = - \tan \frac{A+B}{2} \tan \frac{B+C}{2} \tan \frac{A}{2} \tan \frac{C}{2}. \tag{15} \]
By symmetry \((B \leftrightarrow D)\), we also have
\[ \frac{\rho_c \rho_d}{R_c R_d} = - \tan \frac{A+D}{2} \tan \frac{D+C}{2} \tan \frac{A}{2} \tan \frac{C}{2}. \tag{16} \]
Now using the sum of angles in a quadrilateral,
\[ \tan \frac{A+B}{2} = - \tan \frac{D+C}{2} \]
and
\[ \tan \frac{B+C}{2} = - \tan \frac{A+D}{2}. \]
Hence
\[ \tan \frac{A+B}{2} \tan \frac{B+C}{2} = \tan \frac{A+D}{2} \tan \frac{D+C}{2} \]
so by (15) and (16) we have
\[ \frac{\rho_a \rho_b}{R_a R_b} = \frac{\rho_c \rho_d}{R_c R_d}. \tag{17} \]
The quadrilateral is extangential if and only if the internal angle bisectors at \(A\) and \(C\), and the external angle bisectors at \(B\) and \(D\) are concurrent, which is equivalent to \(E_a \equiv E_b \equiv E_c \equiv E_d\). This in turn is equivalent to that \(\rho_a = \rho_b = \rho_c = \rho_d\). Hence by (17) the quadrilateral is extangential if and only if \(R_a R_b = R_c R_d\).

The second condition \(R_a R_d = R_b R_c\) is proved in the same way. \(\square\)
We have not found a way to express the exradius (the radius in the excircle) in terms of the escribed radii in comparison to Corollary 6. Instead we have the following formulas, which although they are simple, we cannot find a reference for. They resemble the well known formulas $r = \frac{K}{a+c} = \frac{K}{b+d}$ for the inradius in a tangential quadrilateral with sides $a$, $b$, $c$, $d$ and area $K$.

**Theorem 8.** An extangential quadrilateral with sides $a$, $b$, $c$ and $d$ has the exradius

$$\rho = \frac{K}{|a-c|} = \frac{K}{|b-d|}$$

where $K$ is the area of the quadrilateral.

**Proof.** We prove the formulas in the case that is shown in Figure 8. The area of the extangential quadrilateral $ABCD$ is equal to the areas of the triangles $ABE$ and $ADE$ subtracted by the areas of $BCE$ and $CDE$. Thus

$$K = \frac{1}{2}a\rho + \frac{1}{2}d\rho - \frac{1}{2}b\rho - \frac{1}{2}c\rho = \frac{1}{2}\rho(a + d - b - c)$$

where the exradius $\rho$ is the altitude in all four triangles. Hence

$$\rho = \frac{2K}{a - c + d - b} = \frac{K}{a - c} = \frac{K}{d - b}$$

since here we have $a + b = c + d$ (the excircle is outside of $C$), that is $a - c = d - b$. To cover all cases we put absolute values in the denominators. \qed

![Figure 8. Calculating the area of $ABCD$ with four triangles](image)

This theorem indicates that the exradii in all parallelograms (and hence also in all rhombi, rectangles and squares) are infinite, since in all of them $a = c$ and $b = d$. 
Similar metric characterizations of tangential and extangential quadrilaterals

References


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