A Triad of Circles Tangent Internally to the Nine-Point Circle

Nikolaos Dergiades and Alexei Myakishev

Abstract. Given an acute triangle, we construct the three circles each tangent to two sides and to the nine point circle internally. We show that the centers of these three circles are collinear.

In this note we construct, for a given acute triangle, the three circles each tangent to two sides of the triangle and tangent to the nine point circle internally. We show that the centers of these three circles are collinear ([1, 3]).

Let \( ABC \) be the given triangle with incenter \( I \). For three points \( A', B', C' \) on the respective angle bisectors, write the vectors

\[
\mathbf{IA'} = p\mathbf{IA}, \quad \mathbf{IB'} = q\mathbf{IB}, \quad \mathbf{IC'} = r\mathbf{IC}.
\]

Since

\[
\left( \frac{a}{p} \right) \mathbf{IA'} + \left( \frac{b}{q} \right) \mathbf{IB'} + \left( \frac{c}{r} \right) \mathbf{IC'} = a\mathbf{IA} + b\mathbf{IB} + c\mathbf{IC} = 0,
\]

the three points \( A', B', C' \) are collinear if and only if \( \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0 \).

Now consider the nine-point circle of triangle \( ABC \). This is tangent to the incircle at the Feuerbach point \( F_e \). The power of \( A \) is \( d^2 = \frac{1}{2} S_A \). If we apply
inversion with center $A$ and power $d^2$, the inverse of the incircle is a circle $(A')$ tangent to $AB$, $AC$ and the nine-point circle at the second intersection $F_1$ of the line $AF_e$. We have

$$\frac{AA'}{AI} = \frac{d^2}{(s-a)^2}.$$ 

Hence, $p = \frac{IA'}{IA} = \frac{2(a-b)(a-c)}{(b+c-a)^2}$.

Similarly for the other centers $B'$, $C'$ we have

$$q = \frac{2(b-c)(b-a)}{(c+a-b)^2}, \quad r = \frac{2(c-a)(c-b)}{(a+b-c)^2}.$$ 

It is easy to prove that

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0.$$ 

Therefore, the three centers $A'$, $B'$, $C'$ are collinear.

These centers are

$$A' = pA + (1-p)I = \left(\frac{p(a+b+c)}{1-p} + a : b : c\right),$$

$$B' = qB + (1-q)I = \left(a : \frac{q(a+b+c)}{1-q} + b : c\right),$$

$$C' = rC + (1-r)I = \left(a : b : \frac{r(a+b+c)}{1-r} + c\right).$$

If the line containing these centers has barycentric equation $ux + vy + wz = 0$ with reference to triangle $ABC$, then
A triad of circles tangent internally to the nine-point circle

\[ A' = \left( -\frac{bv + cw}{u} : b : c \right), \quad B' = \left( a : -\frac{au + cw}{v} : c \right), \quad C' = \left( a : b : -\frac{au + bv}{w} \right). \]

It follows that

\[ \frac{p - 1}{p} = \frac{(a + b + c)u}{au + bv + cw}, \quad \frac{q - 1}{q} = \frac{(a + b + c)v}{au + bv + cw}, \quad \frac{r - 1}{r} = \frac{(a + b + c)w}{au + bv + cw}, \]

and

\[ u : v : w = \frac{p - 1}{p} : \frac{q - 1}{q} : \frac{r - 1}{r} = \frac{b^2 + c^2 - a^2}{2(b - c)(c - a)} : \frac{c^2 + a^2 - b^2}{2(a - b)(b - c)} : \frac{a^2 + b^2 - c^2}{2(b - c)(c - a)}. \]

The line containing these points has equation

\[ (b - c)S_Ax + (c - a)S_By + (a - b)S_Cz = 0. \]

This line contains the orthocenter \( \left( \frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C} \right) \) and the Spieker center. As such, it is the Soddy line of the inferior triangle. It is perpendicular to the Gergonne axis, and is the trilinear polar of \( X_{1897} \). Randy Hutson [2] has remarked that this is also the Brocard axis of the excentral triangle.

We conclude with two remarks about the constructions in this note.

1. If angle \( A \) is acute, then the circle \( (A') \) is tangent internally to the nine-point circle, and the circle \( (A'') \) inverse to the \( A \)-excircle is tangent externally to the nine-point circle (see Figure 3).
(2) The constructions apply also to obtuse triangles. If angle $A$ is obtuse, the points $A'$, $A''$ are on the extension of $IA$. The circle $(A')$ is tangent externally to the nine-point circle, and the inverse of the $A$-excircle is a circle tangent internally to the nine-point circle (see Figure 4).

Figure 4.

References


Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Alexei Myakishev: Belomorskaia-12-1-133,Moscow, Russia, 125445
E-mail address: amyakishev@yahoo.com