A Vector-based Proof of Morley’s Trisector Theorem

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Abstract. A proof is given of Morley’s trisector theorem using elementary vector analysis and trigonometry. The known expression for the side of Morley’s equilateral triangle is also obtained.

Since its formulation in 1899, many proofs of Morley’s trisector theorem have appeared, typically based on plane geometry or involving trigonometry; a historical overview of this theorem with numerous references up to the year 1977 can be found in [4]. Some of the more recent geometric proofs are of the “backward” type [2, 5]; a group-theoretic proof was also given [3]. About fifteen different methods that were used to prove Morley’s theorem are described in detail in [1], with comments on their specific characteristics. The website [1] also provides the related references, which span from the year 1909 to 2010.

In this note we prove the theorem in two stages. First a lemma is proved by use of the dot product of vectors and trigonometry, then the theorem itself easily follows from elementary geometry.

Morley’s Theorem. In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.

Let the angles of triangle $ABC$ be of amplitude $3\alpha$, $3\beta$, $3\gamma$, then $\alpha + \beta + \gamma = 60^\circ$. The adjacent trisectors meet to form Morley’s triangle $PQR$; the line extensions of $BR$ and $CQ$ intersect at $D$. In triangle $BDC$ the bisector of angle
$D$ is concurrent with the other two bisectors $BP$ and $CP$ at $P$, the incenter of the triangle. First, a lemma is proved, from which the theorem easily follows.

**Lemma.** The line $DP$ is perpendicular to the line $RQ$.

*Proof.* Here use is made of the vector method in conjunction with trigonometry. Let $e$ be the unit vector along $DP$ and $s_1$ the vector representing the side $QR$ of triangle $PQR$. Then the lemma can be restated as saying that the scalar product $s_1 \cdot e$ vanishes. Figure 1 shows that $s_1 = v_3 - v_2$ so that we must prove that $$(v_3 - v_2) \cdot e = 0.$$  

In triangle $BDC$ we have $2\beta + 2\gamma = 120^\circ - 2\alpha$, therefore $\angle D = 60^\circ + 2\alpha$. This angle is bisected by the line $DP$, hence $\angle QDP = \angle RDP = 30^\circ + \alpha$. By the exterior angle theorem $\angle AQR = \alpha + \gamma$ and $\angle ARD = \alpha + \beta$. The angle between the vectors $v_3$ and $e$, being the difference between angles $RDP$ and $ARD$, is $30^\circ - \beta$. Similarly, the angle between $v_2$ and $e$ is $30^\circ - \gamma$. From these,

$$(v_3 - v_2) \cdot e = v_3 \cdot e - v_2 \cdot e = v_3 \cos(30^\circ - \beta) - v_2 \cos(30^\circ - \gamma) = v_3 \sin(60^\circ + \beta) - v_2 \sin(60^\circ + \gamma).$$  

(1)

The magnitudes of $v_3$ and $v_2$ can be found by applying the law of sines to triangles $ARB$ and $AQC$ respectively:

$$v_3 = \frac{c \sin \beta}{\sin(\alpha + \beta)} = \frac{c \sin \beta}{\sin(60^\circ - \gamma)}, \quad v_2 = \frac{b \sin \gamma}{\sin(\alpha + \gamma)} = \frac{b \sin \gamma}{\sin(60^\circ - \beta)}.$$  

Substituting these expressions into (1), we obtain

$$(v_3 - v_2) \cdot e = \frac{c \sin \beta \sin(60^\circ + \beta)}{\sin(60^\circ - \gamma)} - \frac{b \sin \gamma \sin(60^\circ + \gamma)}{\sin(60^\circ - \beta)}$$

$$= \frac{c \sin \beta \sin(60^\circ + \beta) \sin(60^\circ - \beta) - b \sin \gamma \sin(60^\circ + \gamma) \sin(60^\circ - \gamma)}{\sin(60^\circ - \beta) \sin(60^\circ - \gamma)}$$

$$= \frac{1}{4} \cdot \frac{c \sin 3\beta - b \sin 3\gamma}{\sin(60^\circ - \beta) \sin(60^\circ - \gamma)}$$

with the aid of the identity

$$\sin x \sin(60^\circ + x) \sin(60^\circ - x) = \frac{1}{4} \sin 3x,$$  

(2)

which can be easily proved through the product-to-sum trigonometric formulas.

The law of sines for triangle $ABC$ yields $c \sin 3\beta - b \sin 3\gamma = 0$. Therefore, $$(v_3 - v_2) \cdot e = 0.$$  

**Proof of Morley’s Theorem.** Knowing that $DP \perp RQ$, we see that $DP$ divides $DQR$ into two congruent right triangles (with a common leg and a pair of equal acute angles) so that $DQ = DR$. Consequently, triangles $DPQ$ and $DPR$ are also congruent (by SAS), and $s_2 = s_3$. The whole procedure can be used to prove that $s_1 = s_2$. It follows that $s_1 = s_2 = s_3$, and triangle $PQR$ is equilateral. This completes the proof of Morley’s theorem.
Remark. Since triangle $DQR$ is composed of two congruent right triangles, and $\angle QDR = 30^\circ + \alpha$, its complement $\angle DQR = 60^\circ - \alpha$.

The side of Morley's triangle. The side length $s$ of the equilateral triangle $PQR$ can be calculated by applying the law of sines to triangle $AQR$, whose angles are now known. Since $\angle RAQ = \alpha$, and $\angle AQR = \angle AQD + \angle DQR = (\alpha + \gamma) + (60^\circ - \alpha) = 60^\circ + \gamma$, we find that

$$s = \frac{v_3 \sin \alpha}{\sin(60^\circ + \gamma)} = \frac{c \sin \alpha \sin \beta}{\sin(60^\circ + \gamma) \sin(60^\circ - \gamma)}.$$

By multiplying both terms of the last fraction by $\sin \gamma$, and using in the denominator the identity (2), we get the known expression for the side of Morley’s triangle

$$s = \frac{4c \sin \alpha \sin \beta \sin \gamma}{\sin 3\gamma} = 8R \sin \alpha \sin \beta \sin \gamma,$$

where $R = \frac{c}{2 \sin 3\gamma}$ is the radius of the circumcircle of triangle $ABC$.

References


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