The Touchpoints Triangles and the Feuerbach Hyperbolas

Sándor N. Kiss and Paul Yiu

Abstract. In this paper we generalize the famous Kariya theorem on the perspectivity of a given triangle with the homothetic images of the intouch triangle from the incenter to the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. We also study in some details the triangle formed by the orthocenters of the touchpoints triangles. An elegant construction is given for the asymptotes of the Feuerbach hyperbolas.

1. Introduction

Consider a triangle $ABC$ with incircle $I(r)$ tangent to the sides $BC$, $CA$, $AB$ at $X$, $Y$, $Z$ respectively. These form the intouch triangle $T_i$ of $ABC$. For a real number $t$, let $T_i(t)$ be the image of the $T_i$ under the homothety $h(I, \frac{r}{t})$. Its vertices are the points $X(t)$, $Y(t)$, $Z(t)$ on the lines $IX$, $IY$, $IZ$ respectively, such that $IX(t) = IY(t) = IZ(t) = t$.

The famous Kariya’s theorem asserts that the lines $AX(t)$, $BY(t)$, $CY(t)$ are concurrent, i.e., the triangles $ABC$ and $T_i(t)$ are perspective, and that the perspector is a point $Q(t)$ on the Feuerbach hyperbola $F$, the rectangular circum-hyperbola which is the isogonal conjugate of the line $OI$ joining the circumcenter and the incenter of $ABC$. This fact was actually known earlier to J. Neuberg and H. Mandart; see [5] and the interesting note in [2, §1242]. We revisit in §3 this theorem with a proof leading to simple relations of the perspectors $Q(t)$ and $Q(-t)$ (Proposition 4 below), and their isogonal conjugates on the line $OI$. In §5 we obtain analogous results by replacing $T_i(t)$ by homothetic images of the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. Some properties of the triad of ex-Feuerbach hyperbolas are established in §§7–10. Specifically, we give an elegant construction of the asymptotes of the Feuerbach hyperbolas in §10. The final section §11 is devoted to further properties of the touchpoints triangles, in particular, the loci of perspectors of the triangle $H^aH^bH^c$ formed by their orthocenters.
2. Generalization of Kariya’s theorem

Let \( a, b, c \) be the side lengths of triangle \( ABC \), \( s = \frac{1}{2}(a+b+c) \) the semiperimeter, and \( \Delta \) the area of the triangle. It is well known that

(i) \( \Delta = rs \),
(ii) \( abc = 4R\Delta \) for the circumradius \( R \), and
(iii) \( 16\Delta^2 = a^2(b^2 + c^2 - a^2) + b^2(c^2 + a^2 - b^2) + c^2(a^2 + b^2 - c^2) \).

We shall work with homogeneous barycentric coordinates with reference to triangle \( ABC \), and refer to [7] for basic results and formulas.

Let \( T \) be the pedal triangle of a point \( P = (u : v : w) \) in homogeneous barycentric coordinates. The vertices of \( T \) are the points

\[
X = (0 : (a^2 + b^2 - c^2)u + 2a^2v : (c^2 + a^2 - b^2)u + 2a^2w), \quad Y = ((a^2 + b^2 - c^2)v + 2b^2u : 0 : (b^2 + c^2 - a^2)v + 2b^2w), \quad Z = ((c^2 + a^2 - b^2)w + 2c^2u : (b^2 + c^2 - a^2)w + 2c^2v : 0).
\]

For a real number \( k \), let \( T_k \) be the image of \( T \) under the homothety \( h(P, k) \).

**Lemma 1.** The vertices of \( T_k \) are the points

\[
X_k = (2a^2(1-k)u : (a^2 + b^2 - c^2)ku + 2a^2v : (c^2 + a^2 - b^2)ku + 2a^2w), \quad (1)
Y_k = ((a^2 + b^2 - c^2)kv + 2b^2u : (b^2 + c^2 - a^2)kv + 2b^2w), \quad (2)
Z_k = ((c^2 + a^2 - b^2)kw + 2c^2u : (b^2 + c^2 - a^2)kw + 2c^2v : 2c^2(1-k)w). \quad (3)
\]

**Proof.** The point \( X_k \) divides the segment \( PX \) in the ratio \( PX_k : X_kX = k : 1-k \). In absolute barycentric coordinates,

\[
X_k = (1-k)(u,v,w) + k(0,(a^2 + b^2 - c^2)u + 2a^2v,(c^2 + a^2 - b^2)u + 2a^2w) \quad \frac{2a^2(u + v + w)}{2a^2(u + v + w)}
\]

Ignoring the denominator, we obtain the homogeneous barycentric coordinates of \( X_k \) given above; similarly for \( Y_k \) and \( Z_k \). \( \square \)

**Proposition 2.** The only points that satisfy Kariya’s theorem, as the incenter \( I \) does, are the orthocenter \( H \), the circumcenter \( O \), and the three excenters \( I_a, I_b, I_c \).

**Proof.** The equations of the lines \( AX_k, BY_k, CZ_k \) are

\[
((c^2 + a^2 - b^2)ku + 2a^2w)y - ((a^2 + b^2 - c^2)ku + 2a^2v)z = 0,
-(b^2 + c^2 - a^2)kv + 2b^2w)x + ((a^2 + b^2 - c^2)kv + 2b^2u)z = 0,
(b^2 + c^2 - a^2)kw + 2c^2v)x - ((c^2 + a^2 - b^2)kw + 2c^2u)y = 0.
\]

They are concurrent if and only if

\[
\begin{vmatrix}
0 & (c^2 + a^2 - b^2)ku + 2a^2w & -(a^2 + b^2 - c^2)ku + 2a^2v \\
-(b^2 + c^2 - a^2)kv + 2b^2w & 0 & (a^2 + b^2 - c^2)kv + 2b^2u \\
(b^2 + c^2 - a^2)kw + 2c^2v & -(c^2 + a^2 - b^2)kw + 2c^2u & 0
\end{vmatrix} = 0.
\]
Equivalently, \(2F_1(P) \cdot k - F_2(P) = 0\) for every \(k\), where

\[
F_1(P) = \sum_{\text{cyclic}} a^2(b^2 + c^2 - a^2)u(c^2v^2 - b^2w^2),
\]

\[
F_2(P) = \sum_{\text{cyclic}} (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)u(c^2v^2 - b^2w^2).
\]

This means that \(F_1(P) = F_2(P) = 0\). The point \(P\) is common to the McCay cubic \(pK(X(6), X(3))\), and the orthocubics \(pK(X(6), X(4))\). These appear in [3] as \(K003\) and \(K006\) respectively. It is known that the common points of these circumcubics are the vertices of \(ABC\) and the points \(H, O, I, I_a, I_b, I_c\), as can be readily verified. \(\square\)

The case \(P = H\) is trivial because the perspector is \(H\) for every \(k\).

The case \(P = O\) is also trivial because the triangle \(T_k\) is homothetic to \(ABC\), and the perspector is obviously the point \(Q_k\) on the Euler line dividing \(OH\) in the ratio \(k : 2\). This follows from

\[
OQ_k : Q_kH = OX(k) : HA = k : 2.
\]

In the remainder of this paper, we study the cases when \(P\) is the incenter or an excenter.
3. Kariya’s theorem and the Feuerbach hyperbola

The pedal triangle of the incenter $I = (a : b : c)$ has vertices

\[
X = (0 : a + b - c : c + a - b),
Y = (a + b - c : 0 : b + c - a),
Z = (c + a - b : b + c - a : 0).
\]

The coordinates of $X(t), Y(t), Z(t)$ can be determined from equations (1), (2), (3) by putting $k = \frac{t}{t}$.

**Proposition 3.** The lines $AX(t), BY(t), CZ(t)$ are concurrent at the point

\[
Q(t) = \left( \frac{1}{2rbc + t(b^2 + c^2 - a^2)} : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)} \right),
\]

which is the isogonal conjugate of the point $P(t)$ dividing $OI$ in the ratio $OP(t) : P(t)I = R : t$.

**Proof.** Writing the homogeneous barycentric coordinates of $X(t), Y(t), Z(t)$ as

\[
X(t) = \left( **:** : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)} \right),
Y(t) = \left( \frac{1}{2rbc + t(b^2 + c^2 - a^2)} : **:** : \frac{1}{2rca + t(c^2 + a^2 - b^2)} \right),
Z(t) = \left( \frac{1}{2rbc + t(b^2 + c^2 - a^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)} : **:** \right),
\]

we note that the lines $AX(t), BY(t), CZ(t)$ are concurrent at a point $Q(t)$ with coordinates given in (4) above. This is clearly the isogonal conjugate of the point

\[
P(t) = (2rabc \cdot a + ta^2(b^2 + c^2 - a^2), 2rabc \cdot b + tb^2(c^2 + a^2 - b^2),
2rabc \cdot c + tc^2(a^2 + b^2 - c^2))
\]

\[
= 2rabc(a + b + c) + t((a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2))
\]

\[
= 2r \cdot 4Rrs(a + b + c) \cdot I + t \cdot 16\Delta^2 \cdot O
\]

\[
= 16r^2s^2 \cdot R \cdot I + 16\Delta^2 \cdot t \cdot O
\]

\[
= 16r^2s^2(R \cdot I + t \cdot O).
\]

In absolute barycentric coordinates, $P(t) = \frac{R \cdot I + t \cdot O}{R + t}$. This is the point dividing $OI$ in the ratio $OP(t) : P(t)I = R : t$.

It follows that the locus of the point $Q(t)$ is the Feuerbach hyperbola $\mathcal{F}$. The center is the Feuerbach point $F_\infty$, the point of tangency of the incircle and the nine-point circle (see Figure 2). Note that for each $t$, the points $P(t)$ and $P(-t)$ divide $OI$ harmonically.
Proposition 4. The line joining $Q(t)$ and $Q(-t)$ contains the triangle center

$$H_i = \left( \frac{a(b+c)}{b+c-a} : \frac{b(c+a)}{c+a-b} : \frac{c(a+b)}{a+b-c} \right).$$

Proof. The line joining $Q(t)$ to $Q(-t)$ has equation

$$\sum_{\text{cyclic}} a(b-c)(b+c-a)(4b^2c^2r^2 - t^2(b^2 + c^2 - a^2)^2)x = 0.$$ 

With $(x : y : z)$ given in (5) we have

$$\sum_{\text{cyclic}} a^2(b^2-c^2)(4b^2c^2r^2 - t^2(b^2 + c^2 - a^2)^2)$$

$$= 4a^2b^2c^2r^2 \left( \sum_{\text{cyclic}} (b^2-c^2) \right) - t^2 \left( \sum_{\text{cyclic}} a^2(b^2-c^2)(b^2 + c^2 - a^2)^2 \right)$$

$$= 0.$$ 

While the first sum obviously is zero, the second sum vanishes because the Euler line

$$\sum_{\text{cyclic}} (b^2-c^2)(b^2 + c^2 - a^2)x = 0$$

contains the circumcenter. \qed
Remark. The triangle center $H_i$ is the orthocenter of the intouch triangle $T_i$; it appears as $X(65)$ in [4]. It divides $OI$ in the ratio $R \colon r = -r$. Its isogonal conjugate is the Schiffler point

$$S_c = \left( \frac{a(b + c - a)}{b + c} : \frac{b(c + a - b)}{c + a} : \frac{c(a + b - c)}{a + b} \right),$$

which is the point of concurrency of the Euler lines of the four triangles $ABC$, $IBC$, $ICA$, and $IAB$. Therefore, $H_i$ is a point on the Jerabek hyperbola $\mathcal{J}$:

$$\frac{a^2(b^2 - c^2)(b^2 + c^2 - a^2)}{x} + \frac{b^2(c^2 - a^2)(c^2 + a^2 - b^2)}{y} + \frac{c^2(a^2 - b^2)(a^2 + b^2 - c^2)}{z} = 0,$$

the isogonal conjugate of the Euler line.

4. The touchpoints triangles

Consider the $A$-excircle $I_a(r_a)$ of triangle $ABC$, tangent to the sidelines $BC$ at $X_a$, $CA$ at $Y_a$, and $AB$ at $Z_a$ respectively. We call triangle $T_a := X_aY_aZ_a$ the $A$-touchpoints triangle. Clearly,

$$I_aX_a = I_aY_a = I_aZ_a = r_a = \frac{\Delta}{s - a}.$$

In homogeneous barycentric coordinates these are the points

$$I_a = (-a : b : c),$$

$$X_a = (0 : c + a - b : a + b - c),$$

$$Y_a = (-c + a - b : 0 : a + b + c),$$

$$Z_a = (-a + b - c : a + b + c : 0).$$

Similarly, we also have the $B$-touchpoints triangle $T_b := X_bY_bZ_b$ from the $B$-excircle $I_b(r_b)$ and the $C$-touchpoints triangles $T_c := X_cY_cZ_c$ from the $C$-excircle $I_c(r_c)$ (see Figure 3).

Consider the reflection $I'_a$ of $I_a$ in the line $Y_aZ_a$. Since $Y_aZ_a$ is perpendicular to the line $AI_a$, $I'_a$ lies on $AI_a$, and $I_aI'_a = 2r_a \sin \frac{A}{2}$. On the other hand, $I_aI$, being a diameter of the circle through $I$, $B$, $I_a$, $C$, has length $\frac{a}{\sin \left(\frac{A}{2} + \frac{B}{2}\right)} = \frac{a}{\cos \frac{A}{2}}$.

It follows that

$$I_aI'_a : I_aI = 2r_a \sin \frac{A}{2} : \frac{a}{\cos \frac{A}{2}} = r_a : \frac{a}{\sin A} = r_a : 2R = I_aX_a : I_aI'.$$

Therefore, $X_aI'_a$ and $I'I$ are parallel. Note that the midpoint of $X_aI'_a$ is the nine-point center of $T_a$.

The same conclusions apply to the other two touchpoints triangles $T_b := X_bY_bZ_b$ and $T_c := X_cY_cZ_c$ associated with the $B$- and $C$-excircles $I_b(r_b)$ and $I_c(r_c)$.

Corollary 5. (a) The Euler lines of the touchpoints triangles of the excircles are concurrent at $O$.

(b) The nine-point centers of the touchpoints triangles form a triangle perspective with the extouch triangle $X_aY_bZ_c$ at the infinite point of the $OI$ line.
Let $H^a$ be the orthocenter of $T_a$. Since the nine-point center $N_a$ is also the midpoint of its circumcenter $I_a$ and $H^a$, we obtain, from the parallelogram $H^a X_a I_a I'_a$,

\[
H^a = X_a + I'_a - I_a
\]

\[
= X_a + \frac{r_a}{2R} (I - I_a)
\]

\[
= X_a + \frac{2s(s - b)(s - c)}{abc} (I - I_a)
\]

\[
= \frac{(s - b)B + (s - c)C}{a} + \frac{2s(s - b)(s - c)}{abc} \left( \frac{aA + bB + cC}{2s} - \frac{-aA + bB + cC}{2(s - a)} \right)
\]

\[
= \frac{(b + c)(s - b)(s - c)}{bc(s - a)} A + \frac{s(c - a)(s - b)}{ac(s - a)} B - \frac{s(a - b)(s - c)}{ab(s - a)} C.
\]

**Proposition 6.** In homogeneous barycentric coordinates, the orthocenters of the touchpoints triangles are the points

\[
H^a = \left( \frac{a(b + c)}{a + b + c} : \frac{b(c - a)}{a + b - c} : \frac{-c(a - b)}{c + a - b} \right),
\]

\[
H^b = \left( \frac{-a(b - c)}{a + b - c} : \frac{b(c + a)}{a + b + c} : \frac{c(a - b)}{b + c - a} \right),
\]

\[
H^c = \left( \frac{a(b - c)}{c + a - b} : \frac{-b(c - a)}{b + c - a} : \frac{c(a + b)}{a + b + c} \right).
\]

**Remark.** The orthocenter $H^a$ of $T_a$ divides $OI_a$ in the ratio $R : r_a : r_a$; similarly for $H^b$ and $H^c$. These orthocenters lie on the Jerabek hyperbola $\mathcal{J}$ (see Figure 4).
Proposition 7. The triangle $H^aH^bH^c$ is perspective with the orthic triangle $H_aH_bH_c$ (of $ABC$) at $H_i$.

5. Kariya’s theorem for the $A$-touchpoints triangle

Consider the image of the $A$-touchpoints triangle $T_a$ under the homothety $h\left(I_a, \frac{t}{r_a}\right)$ for a real number $t$. This is the triangle $T_a(t)$ with vertices $X_a(t), Y_a(t), Z_a(t)$ on the lines $I_aX_a, I_aY_a, I_aZ_a$ respectively, such that

$$I_aX_a(t) = I_aY_a(t) = I_aZ_a(t) = t.$$ 

These points can be determined from Lemma 1 by putting $k = \frac{t}{r_a}$. In homogeneous barycentric coordinates, they are

$$X_a(t) = (2(r_a - t)a^2 : -2r_aba + t(a^2 + b^2 - c^2) : -2r_aca + t(c^2 + a^2 - b^2)), \quad (8)$$

$$Y_a(t) = (-2r_aba + t(a^2 + b^2 - c^2) : 2(r_a - t)b^2 : 2r_bca + t(b^2 + c^2 - a^2)), \quad (9)$$

$$Z_a(t) = (-2r_aca + t(c^2 + a^2 - b^2) : 2r_bca + t(b^2 + c^2 - a^2) : 2(r_a - t)c^2). \quad (10)$$
**Proposition 8.** The lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at the point

$$Q_a(t) = \left( \frac{1}{2r_a bc + t(b^2 + c^2 - a^2)} : \frac{1}{-2r_a ca + t(c^2 + a^2 - b^2)} : \frac{1}{-2r_a ab + t(a^2 + b^2 - c^2)} \right),$$

(11)

which is the isogonal conjugate of the point $P_a(t)$ dividing $OI_a$ in the ratio

$$OP_a(t) : P_a(t)I_a = R : -t.$$

![Figure 5.](image)

**Proof.** Rewrite the homogeneous barycentric coordinates of $X_a(t)$, $Y_a(t)$, $Z_a(t)$ as follows:

$$X_a(t) = \left( \ast \ast \ast \ast \ast : \frac{1}{-2r_a ca + t(c^2 + a^2 - b^2)} : \frac{1}{-2r_a ab + t(a^2 + b^2 - c^2)} \right),$$

$$Y_a(t) = \left( \frac{1}{2r_a bc + t(b^2 + c^2 - a^2)} : \ast \ast \ast \ast \ast : \frac{1}{-2r_a ab + t(a^2 + b^2 - c^2)} \right),$$

$$Z_a(t) = \left( \frac{1}{2r_a bc + t(b^2 + c^2 - a^2)} : \frac{1}{-2r_a ca + t(c^2 + a^2 - b^2)} : \ast \ast \ast \ast \ast \right).$$

It follows easily that the lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at a point with coordinates given in (11) above (see Figure 5). This is clearly the isogonal
conjugate of the point
\[ P_a(t) = (a^2(2r_a bc + t(b^2 + c^2 - a^2)), b^2(-2r_a ca + t(c^2 + a^2 - b^2)), c^2(-2r_a ab + t(a^2 + b^2 - c^2))) \]
\[ = -2r_a abc(a - b, b, a) + t((a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)) \]
\[ = -2r_a \cdot 4Rrs(b + c - a) \cdot I_a + t \cdot 16\Delta^2 c \cdot O \]
\[ = -16rr_a(s - a) \cdot R \cdot I_a + 16\Delta^2 t \cdot O \]
\[ = -16\Delta^2(R \cdot I_a - t \cdot O). \]

In absolute barycentric coordinates, \( P_a(t) = \frac{R \cdot I_a - t \cdot O}{R - t}. \) This is the point dividing \( OI_a \) in the ratio \( OP_a(t) : P_a(t)I_a = R : -t. \)

**Proposition 9.** The locus of the point \( Q_a(t) \) is the rectangular circum-hyperbola
\[ \mathcal{F}_a : a(b - c)(a + b + c)yz + b(c + a)(a + b - c)zx - c(a + b)(c + a - b)xy = 0 \]
with center
\[ F_a = (-b - c)^2(a + b + c) : (c + a)^2(a + b - c) : (a + b)^2(c + a - b), \]
the point of tangency of the nine-point circle with the A-excircle.

**Proof.** By Proposition 8, the locus of \( Q_a(t) \) is the isogonal conjugate of the line \( OI_a. \) It is a rectangular hyperbola since it contains the orthocenter \( H, \) the isogonal conjugate of \( O. \) The center of the hyperbola is a point on the nine-point circle. The equation of the line \( OI_a \) is
\[ a^2(b^2 + c^2 - a^2) \quad b^2(c^2 + a^2 - b^2) \quad c^2(a^2 + b^2 - c^2) \]
After simplification, this becomes
\[ -bc(b - c)(a + b + c)(b + c - a)x \]
\[ -ca(c + a)(a + b - c)(b + c - a)y \]
\[ + ab(a + b)(b + c - a)(c + a - b)z = 0. \]
Replacing \((x, y, z)\) by \((a^2yz, b^2zx, c^2xy)\) we obtain the equation of the hyperbola \( \mathcal{F}_a \) given above. Since the center of the circumconic \( pyz + qzx + rxy = 0 \) is the point
\[ (p(q + r - p) : q(r + p - q) : r(p + q - r)), \]
with
\[ p = bc(b - c)(a + b + c), \quad q = ca(c + a)(a + b - c), \quad r = -ab(a + b)(c + a - b), \]
we obtain the center of the hyperbola as the point
\[ F_a = \quad p(q + r - p) : q(r + p - q) : r(p + q - r) \]
\[ = -2abc(b - c)^2(a + b + c): 2abc(c + a)^2(a + b - c): 2abc(a + b)^2(c + a - b) \]
\[ = -(b - c)^2(a + b + c): (c + a)^2(a + b - c): (a + b)^2(c + a - b). \]
This is indeed a point on the line joining $I_a$ to the nine-point center
\[ N = \left( a^2(b^2+c^2) - (b^2-c^2)^2 : b^2(c^2+a^2) - (c^2-a^2)^2 : c^2(a^2+b^2) - (a^2-b^2)^2 \right). \]

It is routine to verify that
\[ 2abc(-a,b,c) + (a^2(b^2+c^2) - (b^2-c^2)^2, b^2(c^2+a^2) - (c^2-a^2)^2, c^2(a^2+b^2) - (a^2-b^2)^2) = (b + c - a)(-b + c + a, c + a)(a + b - c, (a + b)(c + a - b)). \]

Since the coordinate sum in (12) is
\[ 2(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4) \]
\[ = 2(a + b + c)(b + c - a)(c + a - b)(a + b - c) \]
\[ = 32\Delta^2, \]
this is the point dividing $NI_a$ in the ratio
\[ 2abc(b + c - a) : 32\Delta^2 = 2 \cdot 4R\Delta(b + c - a) : 32\Delta^2 = \frac{R}{2} : \frac{2\Delta}{b + c - a} = \frac{R}{2} : r_a, \]
i.e., the point of tangency of the nine-point circle and the $A$-excircle.

**Proposition 10.** The line joining $Q_a(t)$ and $Q_a(-t)$ contains the orthocenter $H^a$ of the $A$-touchpoints triangle $T_a$.

**Proof.** The equation of the line $Q_a(t)Q_a(-t)$ is
\[ a(b - c)(a + b + c)(-4b^2c^2r_a^2 + t^2(b^2 + c^2 - a^2)^2)x \]
\[ + b(c + a)(a + b - c)(-4c^2a^2r_a^2 + t^2(c^2 + a^2 - b^2)^2)y \]
\[ - c(a + b)(c + a - b)(-4a^2b^2r_a^2 + t^2(a^2 + b^2 - c^2)^2)z \]
\[ = 0. \]

Substituting the coordinates of the point $H^a$ given in Proposition 6, we obtain
\[ -4a^2b^2c^2r_a^2 ((b^2 - c^2) + (c^2 - a^2) + (a^2 - b^2)) \]
\[ + t^2(a^2(b^2 - c^2)(b^2 + c^2 - a^2)^2 + b^2(c^2 - a^2)(c^2 + a^2 - b^2)^2 \]
\[ + c^2(a^2 - b^2)(a^2 + b^2 - c^2)^2) \]
\[ = 0, \]
as in the proof of Proposition 4. \qed

**6. The triad of ex-Feuerbach hyperbolas**

We call the hyperbola $\mathcal{F}_a$ in Proposition 9 the $A$-ex-Feuerbach hyperbola. We also consider the triangles $T_b(t) := X_b(t)Y_b(t)Z_b(t)$ and $T_c(t) := X_c(t)Y_c(t)Z_c(t)$. These vertices are the points on the lines $I_bX_b, I_bY_b, I_bZ_b$ satisfying
\[ I_bX_b(t) = I_bY_b(t) = I_bZ_b(t) = t, \]
and \( X_c(t), Y_c(t), Z_c(t) \) on \( I_c X_c, I_c Y_c, I_c Z_c \) satisfying

\[
I_c X_c(t) = I_c Y_c(t) = I_c Z_c(t) = t.
\]

In homogeneous barycentric coordinates,

\[
\begin{align*}
X_b(t) &= (2(r_b - t)a^2 : -2r_bab + t(a^2 + b^2 - c^2) : 2r_bca + t(c^2 + a^2 - b^2)), \\
Y_b(t) &= (-2r_bab + t(a^2 + b^2 - c^2) : 2(r_b - t)b^2 : -2r_bbc + t(b^2 + c^2 - a^2)), \\
Z_b(t) &= (2r_bca + t(c^2 + a^2 - b^2) : -2r_bbc + t(b^2 + c^2 - a^2) : 2(r_b - t)c^2), \\
X_c(t) &= (2(r_c - t)a^2 : 2r_cab + t(a^2 + b^2 - c^2) : -2r_cca + t(c^2 + a^2 - b^2)), \\
Y_c(t) &= (2r_cab + t(a^2 + b^2 - c^2) : 2(r_c - t)b^2 : -2r_cbc + t(b^2 + c^2 - a^2)), \\
Z_c(t) &= (-2r_cca + t(c^2 + a^2 - b^2) : -2r_cbc + t(b^2 + c^2 - a^2) : 2(r_c - t)c^2).
\end{align*}
\]

Clearly there are analogous hyperbolas \( \mathcal{F}_b \) and \( \mathcal{F}_c \) which are isogonal conjugates of the lines \( OI_b \) and \( OI_c \). These hyperbolas have centers

\[
\begin{align*}
F_b &= ((b + c)^2(a + b - c) : -(c - a)^2(a + b + c) : (a + b)^2(b + c - a)), \\
F_c &= ((b + c)^2(c + a - b) : (c + a)^2(b + c - a) : -(a - b)^2(a + b + c)).
\end{align*}
\]

**Remark.** The centers of the triad of ex-Feuerbach hyperbolas, being the points of tangency of the nine-point circle with the excircles, are perspective with \( ABC \) at the outer Feuerbach point

\[
X(12) = \left( \frac{(b + c)^2}{b + c - a} : \frac{(c + a)^2}{c + a - b} : \frac{(a + b)^2}{a + b - c} \right).
\]

![Figure 6.](image)
Proposition 11. The triangle \( X_a(t)Y_b(t)Z_c(t) \) is homothetic to the intouch triangle at the point \( (a(r_a - t) : b(r_b - t) : c(r_c - t)) \), which divides \( OI \) in the ratio \( 2R + r - t : -2r \) (see Figure 6).

7. Some special cases

7.1. \( t = R \). By Corollary 5(b), the point \( P_a(R) \) is the infinite point on the line \( OI_a \). It follows that \( Q_a(R) \) is on the circumcircle. It is the (fourth) intersection of the hyperbola \( \mathcal{F}_a \) with the circumcircle. These points are the reflections of \( H \) in \( F_a, F_b, F_c \) respectively (see Figure 7).

7.2. \( t = 2R \). It is well known that the circumcenter of the excentral triangle is the reflection \( I' \) of \( I \) in \( O \), and is the point of concurrency of the perpendiculars from the excenters to the respective sidelines of triangle \( ABC \) (see Figure 6), and the circumradius is \( 2R \). It follows that the points \( X_a(2R), Y_b(2R), Z_c(2R) \) all coincide with this circumcenter. It follows that the lines \( AQ_a(2R), BQ_b(2R), CQ_c(2R) \) are concurrent at this point. \( t = 2R \) is the only nonzero value of \( t \) for which the triangle \( Q_a(t)Q_b(t)Q_c(t) \) is perspective with \( ABC \).

In this case, both \( Y_c(2R) \) and \( Z_b(2R) \) are the reflection of \( I' \) in the line \( I_bI_c \). We call this \( X' \). The line \( AX' \) is the reflection of \( AI' \) in \( I_bI_c \). Since \( I_bI_c \) is the external bisector of angle \( A \) of triangle \( ABC \), \( AX' \) and \( AI' \) are isogonal lines with respect to this angle. Likewise, we have \( Y' = Z_a(2R) = X_c(2R) \) with \( BY' \), \( BI' \) isogonal with respect to \( B \), and and \( Z' = Y_b(2R) = X_b(2R) \) with \( CZ' \), \( CI' \) isogonal with respect to \( C \). It follows that \( AX', BY', CZ' \) are concurrent at a point \( P \), which is the isogonal conjugate of \( I' \), and lies on the Feuerbach hyperbola \( \mathcal{F} \) (see Figure 8).
8. Second tangents from $O$ to the ex-Feuerbach hyperbolas

The hyperbolas $\mathcal{F}_a$, $\mathcal{F}_b$, $\mathcal{F}_c$ also appear in [1], where they are called the ex-central Feuerbach hyperbolas. Neuberg [5] also mentioned these hyperbolas. The $A$-ex-Feuerbach hyperbola $\mathcal{F}_a$, being the isogonal conjugate of the line $OI_a$, is tangent to the line at $I_a$ (see Figure 9). If the second tangent from $O$ to $\mathcal{F}_a$ touches it at $T_a$, then the line $I_aT_a$ is the polar of $O$ with respect to the hyperbola $\mathcal{F}_a$. This is the line

$$bc(b - c)(a + b + c)(a(b^2 + c^2 - a^2) + (b + c)(c + a - b)(a + b - c))x + ca(c + a)(a + b - c)(c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b))y - ab(a + b)(c + a - b)(b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a))z = 0.$$ 

Apart from the excenter $I_a$, this line intersects the hyperbola $\mathcal{F}_a$ again at

$$T_a = \left( \frac{a}{a(b^2 + c^2 - a^2) + (b + c)(c + a - b)(a + b - c)}, \frac{b}{c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b)}, \frac{c}{b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a)} \right).$$
Similarly, the second tangents from $O$ to $\mathcal{F}_b$ and $\mathcal{F}_c$ (apart from $OI_b$ and $OI_c$) touch these hyperbolas at

$$T_b = \left( \frac{a}{c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b)} : \frac{b}{b(c^2 + a^2 - b^2) + (c + a)(a + b - c)(b + c - a)} : \frac{c}{a(b^2 + c^2 - a^2) - (b + c)(c + a - b)(a + b - c)} \right) ,$$

and

$$T_c = \left( \frac{a}{b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a)} : \frac{b}{a(b^2 + c^2 - a^2) - (b + c)(c + a - b)(a + b - c)} : \frac{c}{c(a^2 + b^2 - c^2) + (a + b)(b + c - a)(c + a - b)} \right) .$$

These three points of tangency form a triangle perspective with $ABC$ at

$$T = (a(a(b^2 + c^2 - a^2) - (b + c)(c + a - b)(a + b - c)) : b(b(c^2 + a^2 - b^2) - (c + a)(a + b - c)(b + c - a)) : c(c(a^2 + b^2 - c^2) - (a + b)(b + c - a)(c + a - b))) .$$
This is the triangle center \( X(46) \) in [4]. It has a number of interesting properties. It divides \( OI \) externally in the ratio \( R + r : -2r \), and can be constructed as the cevian quotient \( H/I \). In other words, it is the perspector of the orthic triangle and the excentral triangle. Therefore, the point \( T_a \), and similarly \( T_b \) and \( T_c \), can be easily constructed as follows.

1. Join \( I_a \) and \( H_a \) to intersect the line \( OI \) at \( T \).
2. Join \( A \) and \( T \) to intersect the hyperbola \( \mathcal{F}_a \) at \( T_a \) (see Figure 9).

### 9. A correspondence between the Euler line and the Feuerbach hyperbola

Let \( P = (u : v : w) \) be an arbitrary point. The lines \( P_Ia, P_Ib, P_Ic \) intersect the respective ex-Feuerbach hyperbolas at

\[
W_a = \left( \frac{(b - c)(a + b + c)}{cw - bw} : \frac{(a + b)(c + a - b)}{aw + cu} : \frac{(a + b)(c + a - b)}{bu + av} \right),
\]
\[
W_b = \left( \frac{(b + c)(a + b - c)}{cw + bw} : \frac{(c - a)(a + b + c)}{aw - cu} : \frac{(a + b)(b + c - a)}{bu + av} \right),
\]
\[
W_c = \left( \frac{(b + c)(c + a - b)}{cw + bw} : \frac{(c + a)(b + c - a)}{aw + cu} : \frac{(a - b)(a + b - c)}{bu - av} \right).
\]
These form a triangle perspective with $ABC$. The perspector is the point
$$W = \left( \frac{b + c}{(b + c - a)(cv + bw)} : \frac{c + a}{(c + a - b)(aw + cu)} : \frac{a + b}{(a + b - c)(bu + av)} \right).$$

**Proposition 12.** The perspector $W$ is on the Feuerbach hyperbola if and only if $P$ lies on the Euler line.

**Proof.** The perspector $W$ is on the Feuerbach hyperbola if and only if its isogonal conjugate $W^*$ lies on the line $OI$ with equation
$$\sum_{\text{cyclic}} bc(b - c)(b + c - a)x = 0.$$ 

By substitution, we have
$$0 = \sum_{\text{cyclic}} bc(b - c)(b + c - a) \cdot \frac{a^2(b + c - a)(cv + bw)}{b + c}$$
$$= abc \sum_{\text{cyclic}} \frac{a(b - c)(b + c - a)^2(cv + bw)}{b + c}$$
$$= \frac{abc}{(b + c)(c + a)(a + b)} \sum_{\text{cyclic}} a(b - c)(c + a)(a + b)(b + c - a)^2(cv + bw).$$

Ignoring the nonzero factor, we have
$$0 = \sum_{\text{cyclic}} a(b - c)(c + a)(a + b)(b + c - a)^2(cv + bw)$$
$$= \sum_{\text{cyclic}} (b(c - a)(a + b)(b + c)(c + a - b)^2 \cdot cu$$
$$+ c(a - b)(b + c)(c + a)(a + b - c)^2 \cdot bu)$$
$$= \sum_{\text{cyclic}} bc(b + c)((c - a)(a + b)(c + a - b)^2 + (a - b)(c + a)(a + b - c)^2)u$$
$$= \sum_{\text{cyclic}} bc(b + c) \cdot (-2a(b - c)(b^2 + c^2 - a^2))u$$
$$= -2abc \sum_{\text{cyclic}} (b + c)(b - c)(b^2 + c^2 - a^2)u.$$

This means that $P = (u : v : w)$ lies on the Euler line (with equation given in (6)). \qed

If $P$ divides $OH$ in the ratio $OP : PH = t : 1 - t$, then the isogonal conjugate of $W$ is the point dividing $OI$ in the ratio $OW^* : W^*I = R^2 \frac{(1 - t)}{t} : rt$. A simple
application of Menelaus’ theorem yields the following construction of \( W^* \). Let \( P' \)
be the inferior of \( P \). Then the line \( F_e P' \) intersects \( OI \) at \( W^* \) (see Figure 11).

![Figure 11.](image)

If we put \( W = (x : y : z) \), then \( P \) is the point with coordinates
\[
(u : v : w) = \left( a \left( \frac{-a(b + c)}{(b + c - a)x} + \frac{b(c + a)}{(c + a - b)y} + \frac{c(a + b)}{(a + b - c)z} \right) : b \left( \frac{a(b + c)}{(b + c - a)x} - \frac{b(c + a)}{(c + a - b)y} + \frac{c(a + b)}{(a + b - c)z} \right) : c \left( \frac{a(b + c)}{(b + c - a)x} + \frac{b(c + a)}{(c + a - b)y} - \frac{c(a + b)}{(a + b - c)z} \right) \right).
\]

10. The asymptotes of the Feuerbach hyperbolas

As is well known, the asymptotes of a rectangular circum-hyperbola which is the
isogonal conjugate of a line through \( O \) are the Simson lines of the intersections of
the line with the circumcircle. For the Feuerbach hyperbola and the ex-Feuerbach
hyperbolas, we give an easier construction based on the fact that the lines joining
the circumcenter to the incenter and the excenters are tangent to the respective
Feuerbach hyperbolas.

**Lemma 13.** Let \( P \) be a point on a rectangular hyperbola with center \( O \). The
tangent to the hyperbola at \( P \) intersects the asymptotes at two points on the circle
with center \( P \), passing through \( O \).

**Proof.** Set up a Cartesian coordinate system with the asymptotes as axes. The
equation of the rectangular hyperbola is \( xy = c^2 \) for some \( c \). If \( P \left( ct, \frac{c}{t} \right) \) is a point
on the hyperbola, the tangent at \( P \) is the line \( \frac{1}{2} \left( \frac{c}{t} x + ct y \right) = c^2 \), or \( \frac{c}{t} x + yt = 2c \). It intersects the asymptotes (axes) at \( X(2ct, 0) \) and \( Y \left( 0, \frac{2c}{t} \right) \). Since \( P \) is the
midpoint of \( XY \), \( PO = PX = PY \). \( \square \)
**Proposition 14.** (a) The lines joining the Feuerbach point $F_c$ to the intersections of the incircle with the line $OI$ are the asymptotes of the Feuerbach hyperbola $\mathcal{F}$.

(b) The lines joining the point $F_a$ to the intersections of the $A$-excircle with the line $OI_a$ are the asymptotes of the $A$-ex-Feuerbach hyperbola $\mathcal{F}_a$; similarly for the hyperbolas $\mathcal{F}_b$ and $\mathcal{F}_c$ (see Figure 12).

11. More on the touchpoints triangles

11.1. *The symmedian points of the touchpoints triangles.* Since the $A$-excircle is the circumcircle of the touchpoints triangle $T_a$, and the lines $BC, CA, AB$ are the tangents at its vertices, the symmedian point of $T_a$ is the point of concurrency of $BY_a, CZ_a,$ and $AX_a$, i.e.,

$$K_a = \frac{-(c + a - b)(a + b - c) : (a + b + c)(c + a - b) : (a + b + c)(a + b - c)}{a + b + c}.$$ 

Note that $K_a$ is a point on the $A$-ex-Feuerbach hyperbola $\mathcal{F}_a$.

The line joining $K_a$ to $I_a$ is the Brocard axis of $T_a$. It has equation

$$(b - c)(a + b + c)^2x + (c + a)(a + b - c)^2y - (a + b)(c + a - b)^2z = 0.$$
**Proposition 15.** (a) The Brocard axes of the touchpoints triangles and the intouch triangle are concurrent at the deLongchamps point \( L \), which is the point on the Euler line of triangle \( ABC \) dividing \( OH \) in the ratio \(-1 : 2\).

(b) The van Aubel lines (joining the orthocenter and the symmedian point) of the touchpoints triangles and the intouch triangle are concurrent at \( H^* \), the isotomic conjugate of the orthocenter of \( ABC \) (see Figure 13).

![Figure 13](image-url)

**Remark.** The intersection of the Euler line with the line \( IG_e \) at the deLongchamps point \( L \) is a well known fact. See [6].

**Proposition 16.** The triangle \( H^aH^bH^c \) is perspective with the cevian triangle of \( Q \) if and only if \( Q \) lies on the line

\[
L^\circ : \frac{(b + c)(b^2 + c^2 - a^2)}{b + c - a} x + \frac{(c + a)(c^2 + a^2 - b^2)}{c + a - b} y + \frac{(a + b)(a^2 + b^2 - c^2)}{a + b - c} z = 0
\]

or the Feuerbach hyperbola \( \mathcal{F} \).

(a) If \( Q \) traverses \( L^\circ \), the perspector traverses the line \( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \).

(b) If \( Q \) is on the Feuerbach hyperbola, the perspector \( P \) lies on the Jerabek hyperbola \( \mathcal{J} \). The line joining \( QP \) passes through the orthocenter \( H \) (see Figure 14).
Proof. Let $Q = (u : v : w)$ with cevian triangle $Q_aQ_bQ_c$ where $Q_a = (0 : v : w)$, $Q_b = (u : 0 : w)$, $Q_c = (u : v : 0)$. The equations of the lines $H^aQ_a$, $H^bQ_b$, $H^cQ_c$ are
\begin{align*}
(a + b + c)(c(a - b)(a + b - c)v + b(c - a)(c + a - b)w)x \\
-a(c + a - b)(a + b - c)(b + c)(wy - vz) = 0, \quad (13) \\
(a + b + c)(a(b - c)(b + c - a)w + c(a - b)(a + b - c)u)y \\
-b(a + b - c)(b + c - a)(c + a)(uz - wx) = 0, \quad (14) \\
(a + b + c)(b(c - a)(c + a - b)u + a(b - c)(b + c - a)v)z \\
-c(b + c - a)(c + a - b)(a + b)(vx - uy) = 0. \quad (15)
\end{align*}

Eliminating $x$, $y$, $z$ from equations (13), (14), (15), we have
\begin{align*}
2abc(a + b + c) & \left( \sum_{\text{cyclic}} (b + c)(c + a - b)(a + b - c)(b^2 + c^2 - a^2)u \right) \\
\left( \sum_{\text{cyclic}} a(b - c)(b + c - a)vw \right) = 0.
\end{align*}

Therefore the lines $H^aQ_a$, $H^bQ_b$, $H^cQ_c$ are concurrent if and only if $Q = (u : v : w)$ lies on the line $\mathcal{L}$ or the Feuerbach hyperbola $\mathcal{F}$. 
Eliminating $u, v, w$ from equations (13), (14), (15), we have
\[32\Delta^2 (bcx + cay + abz) \left( \sum_{cyclic} a^2 (b^2 - c^2)(b^2 + c^2 - a^2)yz \right) = 0.\]

Therefore the locus of the point of concurrency is the union of the line $\mathcal{L}(I) : bcx + cay + abz = 0$ (the trilinear polar of the incenter) and the Jerabek hyperbola $\mathcal{J}$.

Now the line $\mathcal{L}$ contains the point
\[ Q_0 = \left( \frac{(a-b)(b+c-a)^2}{b+c} : \frac{b(c-a)(c+a-b)^2}{c+a} : \frac{c(a-b)(a+b-c)^2}{a+b} \right) \]
as is easily verified. Choosing $Q = (u : v : w)$ to be this point, and solving equations (13), (14), (15), we have the perspector
\[ P_0 = (a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2)) \]
on the line $\mathcal{L}(I)$. Therefore, by continuity, when $Q$ traverses the line $\mathcal{L}$, $P$ traverses $\mathcal{L}(I)$.

On the other hand, if $Q$ lies on the Feuerbach hyperbola, then $P$ lies on the Jerabek hyperbola. If we take $Q$ to be the point
\[ \left( \frac{1}{bc + t(b^2 + c^2 - a^2)} : 1 : 1 \right) \]
on the Feuerbach hyperbola, then $P$ is the point
\[ \left( \frac{a(b+c)(b+c-a)}{(b^2 + c^2 - a^2)(2rbc + t(b^2 + c^2 - a^2))} : \cdots : \cdots \right) \]
on the Jerabek hyperbola. The line joining $Q$ and $P$ contains the orthocenter $H$.

\[ \square \]

Remarks. (1) The triangle center $Q_0$ appears in [4] as $X(1021)$.

(2) The triangle center $P_0$ is the intersection of the lines $bcx + cay + abz = 0$ and $ax + by + cz = 0$. It appears in [4] as $X(661)$.

(3) The line $\mathcal{L}$ can be constructed as the line containing the harmonic conjugates of $I_aH \cap BC$ in $BC$, $I_bH \cap CA$ in $CA$, and $I_cH \cap AB$ in $AB$. It is the trilinear polar of the triangle center $X(29)$.

Proposition 17. The triangle $H^aH^bH^c$ is perspective with the anticevian triangle of $Q$ if and only if $Q$ lies in the orthic axis $\mathcal{L}(H) : (b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z = 0$
or the circumconic $\mathcal{C} : a(b^2 - c^2)yz + b(c^2 - a^2)zx + c(a^2 - b^2)xy = 0$

passing through $I$ and its isotomic conjugate $I^* = (bc : ca : ab)$.

(a) If $Q$ traverses the orthic axis, the perspector traverses the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (again).
(b) If $Q$ is on the circumconic $\mathcal{C}$, the perspector $P$ lies on the Jerabek hyperbola (again). The line $QP$ passes through $H_i = \left( \frac{a(b+c)}{b+c-a} : \cdots : \cdots \right)$, the common point of the two conics (see Figure 15).

![Figure 15](image)

**Proof.** Let $Q = (u : v : w)$ with anticevian triangle $Q^aQ^bQ^c$ where $Q^a = (-u : v : w), Q^b = (u : -v : w), Q^c = (u : v : -w)$. The equations of the lines $H^aQ^a, H^bQ^b, H^cQ^c$ are

\[
\begin{align*}
(a + b + c)(c(a - b)(a + b - c)v + b(c - a)(c + a - b)w)x \\
+ (a + b - c)(-a(b + c)(c + a - b)w + c(a - b)(a + b + c)u)y \\
+ (c + a - b)(b(c - a)(a + b + c)u + a(b + c)(a + b - c)v)z = 0, \\
(a + b - c)(c(a - b)(a + b + c)v + b(c + a)(b + c - a)w)x \\
+ (a + b + c)(a(b - c)(b + c - a)w + c(a - b)(a + b - c)u)y \\
+ (b + c - a)(-b(c + a)(a + b - c)u + a(b - c)(a + b + c)v)z = 0, \\
(c + a - b)(-c(a + b)(b + c - a)v + b(c - a)(a + b + c)w)x \\
+ (b + c - a)(a(b - c)(a + b + c)w + c(a + b)(c + a - b)u)y \\
+ (a + b + c)(b(c - a)(c + a - b)u + a(b - c)(b + c - a)v)z = 0.
\end{align*}
\]

(16) (17) (18)
Eliminating \( x, y, z, \) we have

\[
64abc\Delta^2 \left( \sum_{\text{cyclic}} (b^2 + c^2 - a^2)u \right) \left( \sum_{\text{cyclic}} a(b^2 - c^2)vw \right) = 0.
\]

Therefore the lines \( H^aQ^a, H^bQ^b, H^cQ^c \) are concurrent if and only if \( Q = (u : v : w) \) lies on the orthic axis \( \mathcal{L}(H) \) or the circumconic \( \mathcal{C} \).

Eliminating \( u, v, w \) from equations (16), (17), (18), we have

\[
64\Delta^2 (bcx + cay + abz) \left( \sum_{\text{cyclic}} a^2(b^2 - c^2)(b^2 + c^2 - a^2)yz \right) = 0.
\]

Therefore the locus of the point of concurrency is again the union of the line \( \mathcal{L}(I) : bcx + cay + abz = 0 \) (the trilinear polar of the incenter) and the Jerabek hyperbola \( \mathcal{J} \).

Now the circumconic \( \mathcal{C} \) is the circum-hyperbola which is the isotomic conjugate of the line joining the incenter \( I \) to its isotomic conjugate. Its center is the point

\[
(a(b - c)^2 : b(c - a)^2 : c(a - b)^2).
\]

If we choose \( Q \) to be the point \( \left( \frac{1}{at+bc} : \frac{1}{bt+ca} : \frac{1}{ct+ab} \right) \), then the perspector is the point

\[
\left( \frac{a(b^2 + c^2 - a^2)}{at + bc} : \frac{b(c^2 + a^2 - b^2)}{bt + ca} : \frac{c(a^2 + b^2 - c^2)}{ct + ab} \right)
\]

on the Jerabek hyperbola. In fact, \( \mathcal{C} \) intersects \( \mathcal{J} \) at \( H_1 \), and the line joining \( Q \) and \( P \) passes through \( H_1 \).

\[ \square \]

References

  http://math.fau.edu/Yiu/Geometry.html

Sándor N. Kiss: “Constantin Brâncuşi” Technology Lyceum, Satu Mare, Romania
E-mail address: d.sandor.kiss@gmail.com

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road,
Boca Raton, Florida 33431-0991, USA
E-mail address: yiu@fau.edu