Two More Pairs of Archimedean Circles in the Arbelos

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Abstract. We construct two more pairs of Archimedean circles in the arbelos. One of them is a pair constructed by Floor van Lamoen in another way.

In addition to the two pairs of Archimedean circles associated with the arbelos constructed by Dao Thanh Oai [1], we construct two more pairs. Given a segment $AB$ with an interior point $C$, consider the semicircles $(O), (O_1), (O_2)$ with diameters $AB, AC,$ and $CB,$ all on the same side of $AB.$ The perpendicular to $AB$ at $C$ intersects $(O)$ at $D.$ Let $a$ and $b$ be the radii of the semicircles $(O_1)$ and $(O_2)$ respectively. The Archimedean circles have radii $\frac{ab}{a+b}.$

Theorem 1. Let the perpendiculars to $AB$ at $O_1$ and $O_2$ intersect $(O)$ at $E$ and $F$ respectively. If $AF$ intersects $(O_1)$ at $H$ and $BE$ intersects $(O_2)$ at $K,$ then the circles tangent to $CD$ with centers $H$ and $K$ are Archimedean circles.

Proof. Let $M$ and $N$ be the orthogonal projections of $H$ and $K$ on $CD$ respectively. Since $CH$ and $BF$ are both perpendicular to $AF,$ the right triangles $CHM$ and $FBO_2$ are similar (see Figure 1).

$$\frac{HM}{BO_2} = \frac{CH}{FB} = \frac{AC}{AB} \implies HM = BO_2 \cdot \frac{AC}{AB} = b \cdot \frac{2a}{2a+2b} = \frac{ab}{a+b}.$$ 

Therefore the circle $H(M)$ is Archimedean; similarly for $K(N).$ □

Floor van Lamoen has kindly pointed out that this pair has appeared before in a different construction, as $(K_1)$ and $(K_2)$ in [3] (see also $(A_{25a})$ and $(A_{25b})$ in [4]). We show that $H$ and $K$ are intersections of $(O_1)$ and $(O_2)$ with the mid-semicircle with diameter $O_1O_2.$ It is enough to show that $\angle O_1HO_2 = \angle O_1KO_2 = 90^\circ.$

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In Figure 2, $O_2$ is the midpoint of $BC$, and $BF$, $CH$ are parallel. The parallel through $O_2$ to these lines is the perpendicular bisector of $FH$. This means that $O_2F = O_2H$, and

\[
\angle O_1HO_2 = 180^\circ - \angle O_1HA - \angle O_2HF \\
= 180^\circ - \angle O_1AH - \angle O_2FH \\
= \angle AO_2F = 90^\circ.
\]

Similarly, $\angle O_1KO_2 = 90^\circ$.

**Theorem 2.** Let $P$ be the intersection of $AD$ with the semicircle with diameter $AO_2$, and $Q$ that of $BD$ with the semicircle with diameter $BO_1$. The circles tangent to $CD$ with centers $P$ and $Q$ are Archimedean.

![Figure 3](image1)

**Proof.** Let $X$ and $Y$ be the orthogonal projections of $P$ and $Q$ on $CD$ (see Figure 3). Since $BD$ and $O_2P$ are both perpendicular to $AD$, they are parallel.

\[
\frac{PX}{AC} = \frac{DP}{DA} = \frac{BO_2}{BA} \implies PX = AC \cdot \frac{BO_2}{BA} = 2a \cdot \frac{b}{2a + 2b} = \frac{ab}{a + b}.
\]

Therefore, the circle $P(X)$ is Archimedean; similarly for $Q(Y)$. \qed

We show that $PQ$ is a common tangent to the semicircles with diameters $AO_2$ and $BO_1$ (see [5]). In Figure 4, these two semicircles intersect at a point $Z$ on $CD$ satisfying $CZ^2 = 2a \cdot b = a \cdot 2b$. Now, $DP \cdot DA = DZ(DC + ZC) = DQ \cdot DB$. It follows that $\frac{DP}{DQ} = \frac{DB}{DA}$, so that the right triangles $DPQ$ and $DBA$ are similar. Now, if $O_1'$ is the midpoint of $AO_2$, then

\[
\angle O_1'PA = 180^\circ - \angle O_1'PA - \angle DPA \\
= 180^\circ - \angle BAD - \angle DAB \\
= \angle ADB = 90^\circ.
\]

Therefore, $PQ$ is tangent to the semicircle on $AO_2$ at $P$. Similarly, it is also tangent to the semicircle on $BO_1$ at $Q$. It is a common tangent of the two semicircles.
References


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