A Special Point in the Arbelos Leading to a Pair of Archimedean Circles

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Abstract. In 2011 Quang Tuan Bui found a beautiful and simple pair of Archimedean circles, which were published on a website. From this pair we find a special point in the Arbelos leading to a related pair of Archimedean circles.

Figure 1.

Consider an arbelos with \((O)\) being the semicircle with diameter \(AB\), while the point \(C\) on \(AB\) defines the smaller semicircles \((O_1)\) and \((O_2)\) on \(AC\) and \(BC\) respectively. Let the perpendiculars to \(AB\) from \(O_1\) and \(O_2\) meet \((O)\) in \(D\) and \(E\) respectively. The segments \(DA\) and \(DC\) meet \((O_1)\) in two points \(F\) and \(G\). The segment \(FG\) is the diameter of an Archimedean circle (see Figure 1). Likewise an Archimedean circle is found from \(E\).

To prove the correctness of the finding of Bui, we let \(r, r_1\) and \(r_2\) be the radii of \((O), (O_1)\) and \((O_2)\) respectively. Note that \(AF : AD = r_1 : r\), so that \(AD : FD = r : r_2\). Of course \(G\) divides \(CD\) in the same ratio. So, by similarity \(FG = \frac{r_2}{r} \cdot AC = 2 : \frac{r_1 + r_2}{r}\), the Archimedean diameter.

Now one may wonder what the locus of points \(P\) is such that \(PA\) and \(PC\) cut a chord \(ST\) off \((O_1)\) congruent to \(FG\). See Figure 2.

For \(ST\) to be congruent to \(FG\), it is clear that arcs \(FS\) and \(GT\) must be congruent. From this the angles \(DAP\) and \(DCP\) must be congruent, and we conclude that \(ACDP\) is cyclic. The locus of \(P\) is thus the circumcircle of triangle \(ACD\).
Similarly, the locus of $P$ for $PB$ and $PC$ to cut congruent to the one cut out by $EB$ and $EC$ is the circumcircle $BCE$. Now the circumcircles of $ACD$ and $BCE$ intersect, apart from $C$, in a point $L$. This point is thus the only point leading to an Archimedean circle on each of the semicircles $(O_1)$ and $(O_2)$. A notable point (see Figure 3).

References


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