Optimal Packings of Two Ellipses in a Square

Thierry Gensane and Pascal Honvault

Abstract. For each real number $E$ in $[0, 1]$, we describe the densest packing $P_E$ of two non-overlapping congruent ellipses of aspect ratio $E$ in a square. We find three different patterns as $E$ belongs to $[0, 1/2]$, $[1/2, E_0]$ where $E_0 = \sqrt{(6\sqrt{3} - 3)/11}$, and $[E_0, 1]$. The technique of unavoidable sets – used by Friedman for proving the optimality of square packings – allows to prove the optimality of each packing $P_E$.

1. Introduction

We consider the following generalization of the classical disk packing problem in a compact convex domain $K$: Let $n \in \mathbb{N}$ and $E \in [0, 1]$, what is the densest packing of $n$ non-overlapping congruent ellipses of aspect ratio $E$ in $K$?

In this paper, we describe for each aspect ratio $E$ in $[0, 1]$, the densest packing $P_E$ of two congruent unit ellipses of aspect ratio $E$ in the square $K = [0, 1]^2$ and we prove the optimality of these packings. In Figure 1, we display six representative optimal packings of two congruent ellipses. For $E = 1$, the optimal packing $P_1$ is composed of two disks lying in opposite corners, see [4] for a large list of dense packings of congruent disks in the square. An introductory bibliography on disk packing problems can be found in [1, 3]. When $E$ decreases from 1 to $E_0 = \sqrt{(6\sqrt{3} - 3)/11} \approx 0.8198$, the ellipses of optimal packings $P_E$ flatten by keeping a constant tilted angle equals to $-\pi/4$. For $E \in [1/2, E_0]$, the angle of the two ellipses of $P_E$ decreases and when $E = 1/2$ the ellipses reach a third side of the square. When $E$ decreases from $1/2$ to 0, the ellipses slide along the sides and move towards the diagonal.

In all the following we consider only unit ellipses that is, ellipses whose equation is $x^2 - y^2/E^2 = 1$ when their major and minor axes coincide with the cartesian axes. We can reformulate our problem: What is the side length $s_2(E)$ of the smallest square which contains two non-overlapping unit ellipses of aspect ratio $E$?

In order to prove the optimality of square packings, Friedman [2] used sets of unavoidable points. We adapt his definition to the case of ellipse packings: Let $E \in [0, 1]$ and let $P$ be a set of $n - 1$ points in the square $K_s = [0, s]^2$ with $s > 0$. We say that $P$ is a set of unavoidable points in $K_s$ if any unit ellipse of aspect ratio $E$ in $K_s$ contains an element of $P$ (possibly on its boundary). If $P$ is a set of unavoidable points in $K_s$, then $s_{n}(E) \geq s$. For the convenience of the reader, we recall the proof given by Friedman: Shrinking the square $K_s$ by a factor of $1 - \varepsilon/s$ gives a set $P'$ of $n - 1$ points in a square $K_{s-\varepsilon}$ so that any unit ellipse in $K_{s-\varepsilon}$ contains an element of $P'$ in its interior. Therefore no more than
In the case of \( n = 2 \) ellipses and in order to get \( s_2(E) \geq s \), it suffices to show that the center \( \Omega \) of \( K_s = [0, s]^2 \) belongs to each unit ellipse \( e \subset K_s \) of aspect ratio \( E \). In fact, we will consider only unit ellipses \( e_\alpha = e(\lambda, \mu), \alpha, E \) centered at \( (\lambda, \mu) \) with \( \lambda > 0, \mu > 0 \), tilted at an angle \( \alpha \in [-\pi/2, -\pi/4] \) and which are tangent to the axes \( x = 0 \) and \( y = 0 \):

**Fact 1.** Let \( K_s = [0, s]^2 \) be a square of side length \( s \) and \( \Omega = (s/2, s/2) \) its center. If for all \( \alpha \in [-\pi/2, -\pi/4] \), the ellipse \( e_\alpha \) contains the point \( \Omega \), then all unit ellipses \( e \) included in \( K_s \) contain \( \Omega \).

This fact is trivially obtained by contraposition (if a unit ellipse \( e \subset K_s \) does not contain the point \( \Omega \), we apply a translation followed by a reflection or a rotation and we get an ellipse \( e_\alpha \) with \( \alpha \in [-\pi/2, -\pi/4] \) which does not contain the point \( \Omega \)). As we want to find the minimal value of \( s \) such that each ellipse \( e_\alpha \subset K_s \) contains the center \( \Omega \), we consider the intersection points \( I = (x_I(\alpha), x_I(\alpha)) \) and \( J \) of the diagonal \( y = x \) and the ellipse \( e_\alpha \), the abscissa of \( I \) being larger than the one of \( J \). In Section 2 and 3 we will prove that:

- If \( 0 < E \leq 1/2 \), there exists a unique \( \alpha_0 \in [-\pi/2, -\pi/4] \) such that \( x_I(\alpha_0) = \mu(\alpha_0) = \mu_0 \). The center \( \Omega = (\mu_0, \mu_0) \) is an unavoidable point in \( K_{2\mu_0} \) and then \( s_2(E) \geq 2\mu_0 \). The square \( K_{2\mu_0} \) is displayed on the right hand side of Figure 2.
- If \( 1/2 < E < E_0 \), the abcissa \( x_I(\alpha) \) has a minimum value for a unique \( \alpha_0 \in [-\pi/2, -\pi/4] \). The center \( \Omega = (x_I(\alpha_0), x_I(\alpha_0)) \) is an unavoidable point

\( n - 1 \) non-overlapping unit ellipses can be packed into a square of side \( s - \varepsilon \), and \( s_n(E) > s - \varepsilon \). Since this is true for all \( \varepsilon > 0 \), we must have \( s_n(E) \geq s \). The upper bound \( s_n(E) \leq s \) is obtained by constructing a packing of \( n \) non-overlapping unit ellipses in \( K_s \).

Figure 1. Six optimal packings \( P_E \) of two ellipses for \( E = 1, 0.85, 0.69, 0.5, 0.4, 0.05 \).
in $K_{2xI}(\alpha_0)$ and then $s_2(E) \geq 2xI(\alpha_0)$. The three squares in Figure 3 represent $K_{2xI}(\alpha_0)$.

Figure 3. For $\frac{1}{2} < E < E_0$ and if $\alpha \neq \alpha_0$, the center of $K_{2xI}(\alpha_0)$ belongs to the interior of $e_\alpha$.

- If $E_0 \leq E \leq 1$, the abscissa $xI(\alpha)$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$. The center $\Omega = (x_I(-\pi/4), x_I(-\pi/4))$ is an unavoidable point in $K_{2xI}(-\pi/4)$ and then $s_2(E) \geq 2xI(-\pi/4)$, see Figure 4 in Section 3.

We finish the paper by remarking that among all the optimal packings $\mathcal{P}_E$, the densest optimal packings of two congruent ellipses in the square is $\mathcal{P}_{1/2}$.

2. Technical lemmas.

First we precise the coordinates of the center of the ellipse $e_\alpha$ and the parametrization of $e_\alpha$ used in Lemma 3.

Lemma 1. (a) The coordinates of the center of the ellipse $e_\alpha$ are equal to

$$\lambda = \sqrt{\cos^2 \alpha + E^2 \sin^2 \alpha} \quad \text{and} \quad \mu = \sqrt{\sin^2 \alpha + E^2 \cos^2 \alpha}.$$
The function \( \mu \) is decreasing for \( \alpha \in [-\pi/2, -\pi/4] \) and we have \( \mu(-\pi/2) = 1 \) and \( \mu(-\pi/4) = \sqrt{(1 + E^2)/2} \).

(b) We have

\[
(2\lambda \mu)^2 = 4E^2 + (1 - E^2)^2 \sin^2 2\alpha.
\]

(c) The ellipse \( e_\alpha \) can be parameterized by

\[
e_\alpha(t) = \left( \begin{array}{c} \lambda \cos \phi + i \lambda \sin \phi \\ \mu \end{array} \right).
\]

where the angles \( \varphi \in [-\pi/2, 0] \) and \( \psi \in [-\pi, -\pi/2] \) are the respective arguments of the complex numbers \( \cos \alpha + iE \sin \alpha \) and \( \sin \alpha - iE \cos \alpha \).

Proof. (a) Let us consider the parametrization of the ellipse \( e_\alpha \)

\[
e_\alpha(t) = \left( \begin{array}{c} x(t) \\ y(t) \end{array} \right) = \left( \begin{array}{c} \lambda \cos \alpha - \sin \alpha \\ \sin \alpha \cos \alpha \end{array} \right) \left( \begin{array}{c} \cos t \\ E \sin t \end{array} \right).
\]

Let us recall that the orthoptic curve of \( e_\alpha \), i.e the locus of all points where the curve’s tangents meets at right angles, is the circle centered at \( \omega = (\lambda, \mu) \) with radius \( \sqrt{1 + E^2} \). Since the axes \( x = 0 \) and \( y = 0 \) are orthogonal, the origin \((0, 0)\) belongs to this circle and we have

\[
\lambda^2 + \mu^2 = 1 + E^2.
\]

The ellipse \( e_\alpha \) touches tangentially the axe \( x = 0 \). Therefore for some \( t \) we have \( x(t) = x'(t) = 0 \), which gives

\[
\lambda = -\cos \alpha \cos t + E \sin \alpha \sin t,
\]

\[
0 = -\cos \alpha \sin t - E \sin \alpha \cos t.
\]

By adding the squares of equations (6-7), we find \( \lambda^2 = \cos^2 \alpha E^2 \sin^2 \alpha \). The value of \( \mu \) comes from (5).

(b) We have

\[
(\lambda \mu)^2 = (\cos^2 \alpha + E^2 \sin^2 \alpha)(\sin^2 \alpha + E^2 \cos^2 \alpha)
\]

\[
= E^2(\cos^4 \alpha + \sin^4 \alpha) + (1 + E^4) \cos^2 \alpha \sin^2 \alpha
\]

\[
= E^2(1 - 2 \sin^2 \alpha \cos^2 \alpha) + (1 + E^4) \cos^2 \alpha \sin^2 \alpha,
\]

which gives the result.

(c) By definition of \( \varphi \), we have \( \lambda(\cos \varphi + i \sin \varphi) = \cos \alpha + iE \sin \alpha \). We get

\[
x(t) = \lambda + \cos \alpha \cos t - \sin \alpha E \sin t = \lambda + \lambda \cos(t + \varphi) = 2\lambda \cos^2 \left( \frac{t + \varphi}{2} \right).
\]

The expression of \( y(t) \) is obtained similarly. \( \square \)

In the proof of Lemma 3, we change the variable \( \alpha \) to the variable \( T \) that we now define:

Lemma 2. Let us consider the angles \( \varphi \) and \( \psi \) associated to the ellipse \( e_\alpha \) and defined in Lemma 1. We set \( \delta = \frac{\psi - \varphi}{2} \in [-\pi/4, 0] \) and \( T = -\cot \delta \). Then the real number \( T \) decreases monotonically from 1 at \( \alpha = -\pi/2 \) to \( E \) at \( \alpha = -\pi/4 \).
Proof. The definitions of $\varphi$ and $\psi$ give
\[
\frac{\mu}{\lambda} e^{i(\varphi - \psi)} = \frac{\sin(\alpha) - iE \cos(\alpha)}{\cos \alpha + iE \sin \alpha} = \frac{1}{\lambda^2} \left(\sin \alpha \cos \alpha (1 - E^2) - iE\right).
\]
Hence, for $\alpha \in [-\pi/2, -\pi/4]$ we have
\[
\cos(\psi - \varphi) = \frac{\sin 2\alpha (1 - E^2)}{2\lambda \mu} < 0 \text{ and } \sin(\psi - \varphi) = \frac{-E}{\lambda \mu} < 0.
\]
(8)

By (8) and the formula $\tan u = (1 - \cos 2u) / \sin 2u$, we find
\[
\tan \left(\frac{\psi - \varphi}{2}\right) = \frac{1 - \frac{\sin 2\alpha (1 - E^2)}{2\lambda \mu}}{\frac{-E}{\lambda \mu}} = \frac{2\lambda \mu - 2\alpha (1 - E^2)}{-2E}.
\]
(9)

With Lemma 1 (b) we find
\[
\tan \left(\frac{\psi - \varphi}{2}\right) = \frac{1}{2E} \left(-\sqrt{4E^2 + (1 - E^2)^2 \sin^2 2\alpha + (1 - E^2) \sin 2\alpha}\right).
\]

As $\sin 2\alpha$ decreases monotonically from 0 to $-1$ on the interval $[-\pi/2, -\pi/4]$, we find that $\tan \left(\frac{(\psi - \varphi)}{2}\right)$ decreases from $-1$ to $-1/E$. Thus the real number $T = -1/ \tan \left(\frac{(\psi - \varphi)}{2}\right)$ decreases from 1 to $E$. \hfill \Box

Let us recall that $I = (x_I, x_I)$ is the intersection point of the diagonal $y = x$ and the ellipse $e_\alpha$ with a maximal $x_I$.

**Lemma 3.** Let us consider $E_0 = \sqrt{(6\sqrt{3} - 3)/11} \approx 0.8198$.
(a) If $E \in [E_0, 1]$, $x_I$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$.
(b) If $E \in ]1/2, E_0[$, the function $x_I$ reaches a unique minimal value for a unique real number $\alpha_0 \in [-\pi/2, -\pi/4]$.
(c) If $E \in [0, 1/2]$, the function $x_I$ is increasing for $\alpha \in [-\pi/2, -\pi/4]$.

**Proof.** First we prove that
\[
x_I(\alpha) = \frac{E}{\sqrt{T}} \left(\sqrt{ET^2 + (1 + E^2)T + E - T^2E}\right),
\]
(10)

where $T = -\cot \delta$ has been defined in Lemma 2. We use the parametrization of the ellipse $e_\alpha$ given by (3) and we look for some $t \in [0, 2\pi]$ such that
\[
\sqrt{\lambda} \cos \left(\frac{t + \varphi}{2}\right) = \varepsilon \sqrt{\mu} \cos \left(\frac{t + \psi}{2}\right),
\]
(11)

where $\varepsilon = \pm 1$. Since the point $I$ occurs for some $t \in [0, \pi]$ (and $J$ for some $t \in [\pi, 2\pi]$), we have $(t + \varphi)/2 \in [-\pi/4, \pi/2]$, $(t + \psi)/2 \in [-\pi/2, \pi/4]$ and we get $\varepsilon = +1$ because the two cosines in (11) are positive. In this equality we expand $\cos((t + \psi)/2) = \cos((t + \varphi)/2 + \delta)$ and we find
\[
\sqrt{\lambda} \cos \left(\frac{t + \varphi}{2}\right) = \sqrt{\mu} \left(\cos \left(\frac{t + \varphi}{2}\right) \cos \delta - \sin \left(\frac{t + \varphi}{2}\right) \sin \delta\right),
\]
which gives
\[ \tan \left( \frac{t + \varphi}{2} \right) = \frac{\sqrt{\mu \cos \delta} - \sqrt{\lambda}}{\sqrt{\mu \sin \delta}} \]
and
\[ x_I = 2\lambda \cos^2 \left( \frac{t + \varphi}{2} \right) = \frac{2\lambda}{1 + \tan^2 (\frac{t + \varphi}{2})} = \frac{2\lambda \mu \sin^2 \delta}{(2\lambda \mu)^2} \]
By (8) and Lemma 1 (b) and since \( \sin^2 2\delta = 1 - \cos^2 (\psi - \varphi) \), we have
\[ \sin^2 2\delta = \frac{(2\lambda \mu)^2 - (1 - E^2)^2 \sin^2 2\alpha}{(2\lambda \mu)^2} = \frac{4E^2}{(2\lambda \mu)^2}. \]
The previous equality gives \( 2\lambda \mu = -2E / \sin 2\delta \) and by (5) we have \( \lambda + \mu = (1 + E^2 + 2\lambda \mu)^{1/2} \). Substituting these values in (12) we find
\[ x_I = \frac{-E \sin \delta}{\cos \delta \left( \sqrt{1 + E^2 - \frac{E}{\sin \delta \cos \delta}} - \sqrt{\frac{-2E \cos \delta}{\sin \delta}} \right)} = \frac{E}{\sqrt{T} \left( \sqrt{(1 + E^2)T + \frac{E}{\sin^2 \delta}} - \sqrt{T \sqrt{T} \sqrt{2E}} \right)}. \]
It remains to use \( 1 / \sin^2 \delta = 1 + T^2 \) and we get (10). Let us denote by \( f(T) \) the denominator of the right hand side of (10). We find
\[ f'(T) = \frac{g(T) - h(T)}{2\sqrt{T} \sqrt{ET^2 + (1 + E^2)T + E}}, \]
where \( h(T) = 3\sqrt{2E} T \sqrt{ET^2 + (1 + E^2)T + E} \) and \( g(T) = 3ET^2 + 2(1 + E^2)T + E \). Since the functions \( g(T) \) and \( h(T) \) are positive, the sign of \( f'(T) \) is equal to the one of the polynomial \( P(T) = g^2(T) - h^2(T) \), that is
\[ P(T) = -9E^2T^4 - 6E(1 + E^2)T^3 + 4(1 + E^2)T^2 + 4E(1 + E^2)T + E^2. \]
We get \( \lambda''(T) = -36E^2T^3 - 18E(1 + E^2)T^2 + 8(E^4 - E^2 + 1)T + 4E(1 + E^2) \). The discriminant of \( \lambda''(T) = -4(2E^2T^2 + 9E(1 + E^2)T - 2(E^4 - E^2 + 1)) \) is \( \Delta = 16 \cdot 27E^2(11E^4 - 2E^2 + 11) > 0 \). We remark that \( \lambda''(0) > 0 \) and \( \lim_{T \to \infty} \lambda''(T) = -\infty \), then \( \lambda''(T) \) has a unique positive root \( T_2 \). Since \( \lambda''(0) > 0 \) and \( \lim_{T \to \infty} \lambda''(T) = -\infty \), the polynomial \( \lambda''(T) \) has a unique root \( T_1 > T_2 \) and \( \lambda''(T) \geq 0 \) for all \( T \in [0, T_1] \). Finally, \( \lambda''(0) = E^2 > 0 \) implies that the polynomial \( \lambda(T) \) has a unique positive root \( T_0 \).
Moreover, \( P(E) = -E^2(11E^4 + 6E^2 - 9) \) vanishes at a unique positive value \( E_0 = \sqrt{(-3 + 6\sqrt{3})/11} \). We remark that \( T_0 = E_0 \) if and only if \( E = E_0 \). In the three following cases we conclude with Lemma 2:
(a) If \( E \in [E_0, 1] \), we have \( P(E) \leq 0 \). Then \( T_0 \leq E \leq 1 \) which implies that \( P(T) \leq 0 \) for all \( T \in [E, 1] \). So \( f(T) \) is decreasing on \( [E, 1] \) and then \( x_I \) is increasing with respect to \( T \).
(b) If $E \in ]1/2, E_0]$, we have $P(E) > 0$ and $P(1) = 2(E + 1)^2(2E - 1)(E - 2) < 0$. Then $E - T_0 < 1$ which implies that $f(T)$ is increasing on $[E, T_0]$ and decreasing on $[T_0, 1]$. Thus $x_I$ is decreasing for $T \in [E, T_0]$ and increasing for $T \in [T_0, 1]$.

(c) If $E \in ]0, 1/2]$, we have $P(E) \geq 0$ and $P(1) \geq 0$ which implies that $P(T) \geq 0$ on $[E, 1]$. Thus $x_I$ is decreasing for $T \in [E, 1]$.

\[ \square \]

3. Calculation of $s_2(E)$

Now we can describe the various optimal packings of two ellipses in the square and the corresponding side lengths $s_2(E)$.

**Theorem 4.** If $E \leq 1/2$, then

\[ s_2(E) = \sqrt{(1 + E)^2 + \sqrt{(1 + E)^4 - 8E^2}}. \tag{14} \]

The minimum value $s_2(E)$ is obtained for two parallel ellipses $e_1 = e_{\alpha_0}$ and $e_2$ with

\[ \alpha_0 = -\arccos \frac{1}{2} \sqrt{\frac{4 - s_2^2(E)}{1 - E^2}}, \tag{15} \]

and where $e_2$ is the reflection of $e_1$ through the center of the square.

**Proof.** If $\alpha = -\pi/2$, the center $\Omega = (\mu, \mu)$ of $K_{2\mu}$ does not belong to $e_\alpha$ (except for $E = 1/2$). If $\alpha = -\pi/4$, the center $\Omega$ is also the center of the ellipse $e_\alpha$.

We know by Lemma 1 (a) and Lemma 3 (c) that $x_I - \mu$ is increasing for $\alpha \in [-\pi/2, -\pi/4]$. Then there exists a unique angle $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $I = \Omega_0 = (\mu_0, \mu_0)$ with $\mu_0 = \mu(\alpha_0)$. We note that the center $\Omega_0$ belongs to the boundary of the ellipse $e_{\alpha_0}$, see Figure 2. Let us show that in the square $K_{2\mu_0}$, the center $\Omega_0$ is an unavoidable point. By Fact 1, it suffices to show that any ellipses $e_\alpha$ included in $K_{2\mu_0}$ contain $\Omega_0$:

- If $\alpha < \alpha_0$, the ellipse $e_{\alpha}$ is not contained in $K_{2\mu_0}$ because it intersects the upper side $y = 2\mu_0$ (except for $\alpha \in [-\pi/2, -\pi/4]$).

- If $\alpha > \alpha_0$, the point $\Omega_0$ belongs to the interior of the ellipse $e_{\alpha}$ because $x_I$ is increasing for $\alpha \in [\alpha_0, -\pi/4]$.

It remains to calculate $\alpha_0$ and $\mu_0$. First, we show that

\[ \mu_0 - \lambda_0 = \frac{E}{\mu_0}. \tag{16} \]

We have by (4) that $e_{\alpha}(t) = (\mu, \mu)$ if and only if

\[ \begin{align*}
\mu - \lambda &= \cos \alpha \cos t - E \sin \alpha \sin t, \\
0 &= \sin \alpha \cos t + E \cos \alpha \sin t.
\end{align*} \]

Substituting $-E \cos \alpha \sin t / \sin \alpha$ for $\cos t$ in the first equality, we find

\[ \begin{align*}
-E \sin(t) &= (\mu - \lambda) \sin \alpha, \\
\cos t &= (\mu - \lambda) \cos \alpha,
\end{align*} \]

with $E = 0$.

\[ \square \]
which implies \( E^2 = (\mu - \lambda)^2 (\sin^2 \alpha + E^2 \cos^2 \alpha) = (\mu - \lambda)^2 \mu^2 \), and then (16)

since \( \mu \geq \lambda \) for \( \alpha \in [-\pi/2, -\pi/4] \). By (5) we get for \( \alpha = \alpha_0 \),

\[
1 + E^2 + 2\lambda_0 \mu_0 = (\lambda_0 + \mu_0)^2 = (\lambda_0 - \mu_0 + 2\mu_0)^2 = \left(2\mu_0 - \frac{E}{\mu_0}\right)^2.
\]

Since (16) implies \( \mu_0^2 - E = \lambda_0 \mu_0 \), we have

\[
2(\mu_0^2 - E) = 4\mu_0^2 + \frac{E^2}{\mu_0^2} - 4E - (1 + E^2).
\]

This equation in \( \mu_0^2 \) leads to

\[
4\mu_0^2 - (1 + E)^2 + \varepsilon \sqrt{(1 + E)^4 - 8E^2},
\]

where \( \varepsilon = \pm 1 \). The case \( \varepsilon = -1 \) leads to \( 4\lambda_0 \mu_0 = 4\mu_0^2 - 4E = (1 - E)^2 - \sqrt{(1 + E)^4 - 8E^2} \leq 0 \) what is impossible.

Then \( s_2(E) \geq 2\mu_0 = \sqrt{1 + E)^2} + \sqrt{(1 + E)^4 - 8E^2} \). We can pack in \( K_{2\mu_0} \) the reflection of \( e_{\alpha_0} \) through \( \Omega_0 \) and we get the equality (14). We finally obtain the angle (15) by considering \( \mu_0^2 = \sin^2 \alpha_0 + E^2 \cos^2 \alpha_0 \) and \( s_2(E) = 2\mu_0 \). \( \square \)

**Theorem 5.** If \( 1/2 < E < E_0 \), then

\[
s_2(E) = \frac{2E}{\sqrt{T_0 \left( \sqrt{ET_0^2 + (1 + E^2)T_0 + E - T_0\sqrt{2E}} \right)}},
\]

where \( T_0 \) is the unique positive root of (13). The minimum value \( s_2(E) \) is obtained for two parallel ellipses \( e_1 = e_{\alpha_0} \) and \( e_2 \) with

\[
\alpha_0 = -\frac{1}{2} \left( \pi + \arcsin \frac{E(T_0^2 - 1)}{T_0(1 - E^2)} \right),
\]

and where the ellipse \( e_2 \) is the reflection of \( e_1 \) through the center of the square.

**Proof.** We denote by \( \alpha_0 \) the unique angle \( \alpha \) such that \( T_0 = T(\alpha) \) and by \( I_0 \) the intersection of \( e_{\alpha_0} \) with \( y = x \). Since the continuous function \( x_I \) is decreasing for \( \alpha \in [-\pi/2, \alpha_0] \) and increasing for \( \alpha \in [\alpha_0, -\pi/4] \), the point \( I_0 \) belongs to the interior of \( e_{\alpha} \) if \( \alpha \neq \alpha_0 \). Then any ellipse \( e_\alpha \) in \( K_{2x_{I_0}} \) contains \( I_0 \). As Fact 1 gives that \( I_0 \) is an unavoidable point, we have \( s_2(E) \geq 2x_{I_0} \). We can pack the reflection of the ellipse \( e_{\alpha_0} \) through \( I_0 \) and we get \( s_2(E) \leq 2x_{I_0} \). The value of \( x_{I_0} \) is given by (10).

By (2), (9) and since \( -1/T = \tan(\psi - \varphi)/2 \), we have

\[
(2\lambda \mu)^2 = \left( \frac{2E}{T} + (1 - E^2) \sin 2\alpha \right)^2 = 4E^2 + (1 - E^2)^2 \sin^2 2\alpha,
\]

which gives \( \sin 2\alpha = E(T^2 - 1)/(1 - E^2)T \) and (18) for \( 2\alpha \in [-\pi, -\pi/2] \). \( \square \)

**Theorem 6.** If \( E_0 \leq E \leq 1 \), then

\[
s_2(E) = \sqrt{2} \left( \sqrt{1 + E^2} + E \right).
\]
The minimum value \( s_2(E) \) is obtained for two parallel ellipses \( e_1 = e_\alpha \) and \( e_2 \) with \( \alpha = -\pi/4 \) and where \( e_2 \) is the reflection of \( e_1 \) through the center of the square.

**Proof.** We consider again the intersection point \( I_0 \) of the ellipse \( e_{-\pi/4} \) and the diagonal \( y = x \). Since \( x_I \) is decreasing for \( \alpha \in [-\pi/2, -\pi/4] \), any ellipse \( e_\alpha \) in \( K_{2x_{I_0}} \) contains \( I_0 \). As Fact 1 gives that \( I_0 \) is an unavoidable point, we get \( s_2(E) \geq 2x_{I_0} = 2x_I(-\pi/4) = \sqrt{2}(\sqrt{1+E^2} + E) \). As in the two previous cases, we can pack the reflection of the ellipse \( e_{-\pi/4} \) through \( I_0 \) and we get the equality (19). \( \square \)

![Figure 4](image)

Figure 4. For \( E_0 \leq E \leq 1 \) and \( -\pi/4 \leq \alpha < -\pi/4 \), the center of \( K_{2x_I(-\pi/4)} \) belongs to the interior of \( e_\alpha \).

It is not surprising that among all the optimal packings \( \mathcal{P}_E \), the densest one is \( \mathcal{P}_{1/2} \), see Figure 1. We denote by \( d(E) \) the density of \( \mathcal{P}_E \) and we have for all \( E \) in \([0, 1]\),

\[
d(E) = \frac{2\pi E}{s_2^2(E)}.
\]

The formulas (14), (17), (19) for \( s_2(E) \) give that \( d(E) \) equals to

\[
d_1(E) = \frac{2\pi E}{(1+E)^2 + \sqrt{(1+E)^4 - 8E^2}} \quad \text{if} \quad E \in [0, \frac{1}{2}],
\]

\[
d_2(E) = \frac{\pi}{2E} T_0 \left( \sqrt{ET_0^2 + (1+E^2)T_0 + E - T_0\sqrt{2E}} \right)^2 \quad \text{if} \quad E \in \left[\frac{1}{2}, E_0\right[,\n\]

\[
d_3(E) = \frac{\pi E}{\left(\sqrt{1+E^2} + E\right)^2} \quad \text{if} \quad E \in [E_0, 1].
\]

The optimality of \( \mathcal{P}_E \) for all \( E \) on each interval \([0, 1/2], [1/2, E_0[, [E, 1]\) implies the continuity of \( d(E) \) on \([0, 1]\). It is easy to verify that \( d_1(1/2) = d_2(1/2) = \)
\[ \pi/4 \text{ and } d_2(E_0) = d_3(E_0) = \pi E_0/\left(\sqrt{1 + E_0^2} + E_0\right)^2. \] We show that \( d(E) \) is increasing on \([0, 1/2]\) and decreasing on \([1/2, 1]\). For \( E \in [0, \frac{1}{2}] \), we get
\[
d_1'(E) = \frac{2\pi(1 - E^2)}{\sqrt{(E + 1)^4 - 8E^2} \left(1 + 2E + E^2 + \sqrt{(E + 1)^4 - 8E^2}\right)} > 0
\]
and for \( E \in [E_0, 1]\),
\[
d_3'(E) = \frac{\pi\left(\sqrt{1 + E^2} - 2E\right)}{\sqrt{1 + E^2} \left(\sqrt{1 + E^2} + E\right)^2} < 0.
\]
In the case of Theorem 5, we have \( s_2(E) = 2E/f(T_0) \) and \( d_2(E) = (\pi/(2E))f^2(T_0) \).

Since \( T_0 = T_0(E) \) is a single root of (13), \( T_0(E) \) is differentiable at \( E \in [1/2, E_0[ \)
and we get
\[
d_2'(E) = \frac{\pi}{2E^2} \left(2Ef(T_0)f'(T_0)dT_0/dE - f^2(T_0)\right).
\]
As \( f'(T_0) = 0 \), we obtain \( d_2'(E) < 0 \).

References


Thierry Gensane: LMPA J. Liouville, B.P. 699, F-62228 Calais, Univ Lille Nord de France, F-59000 Lille, France
E-mail address: gensane@lmpa.univ-littoral.fr

Pascal Honvault: LMPA J. Liouville, B.P. 699, F-62228 Calais, Univ Lille Nord de France, F-59000 Lille, France
E-mail address: honvault@lmpa.univ-littoral.fr