Generalized Archimedean Arbelos Twins

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Abstract. We generalize the well known Archimedean arbelos twins by extending the notion of arbelos, and we construct an infinite number of Archimedean circles.

1. Archimedean arbelos

On a segment $AB$ we take an arbitrary point $P$ and with diameters $AP$, $PB$, $AB$ we construct the semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, where $R = R_1 + R_2$. If we cut from the large semicircle the small ones then the resulting figure is called from antiquity (Archimedes) arbelos (the shoemaker’s knife). The perpendicular at $P$ to $AB$ meets the large semicircle at $Q$ and divides the arbelos in two mixtilinear triangles with equal incircles.

Theorem 1 (Archimedean arbelos twins). The two circles $K_1(r_1)$ and $K_2(r_2)$ that are inside the arbelos and are tangent to the arbelos and the line $PQ$ have equal radii $r_1 = r_2 = \frac{R_1R_2}{R_1 + R_2}$ (see Figure 1). Equivalently,

$$\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.$$ 

A circle in the arbelos with radius $r_1$ is called an Archimedean circle ([6, p.61]). We can find infinitely many such twin incircles where the above case is a limit special case. We generalize the notion of the arbelos as a triangle whose sides are in general circular arcs. We will investigate some special cases of these arbeloi.

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2. Generalized arbelos

2.1. Soddy type arbelos with 3 vertices tangency points of the arcs. Let $U_X, U_Y, U_Z$ (counter clockwise direction) be the vertices of the arbelos whose sides are arcs of circles tangent to each other at the vertices, with centers $X, Y, Z$ and radii $r_X, r_Y, r_Z$. If the rotation from $U_Y$ to $U_Z$ on the arc with radius $r_X$ is clockwise relative to $U_X$, then $r_X$ is positive, otherwise it is negative. Similarly we characterize the radii $r_Y, r_Z$ relative to the movement on the appropriate arcs from $U_Z$ to $U_X$, and from $U_X$ to $U_Y$. Hence, we can have 8 different cases of arbeloi with the same vertices and the same radii in absolute value. If we change the sign of the radii of the arbelos, then we get the complementary arbelos. Denote by $\Delta$ the area of the triangle $XYZ$ and $r$ the radius of a circle tangent to the arcs-sides of the arbelos.

**Theorem 2.** The radius $r$ of the circle that is tangent to the sides of the above arbelos is given by

\[
\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} + \frac{2\Delta}{r_Xr_Yr_Z},
\]

or

\[
\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} - \frac{2\Delta}{r_Xr_Yr_Z}.
\]

One of these corresponds to the complementary arbelos.

**Proof.** If $K(r)$ is the circle tangent to the sides of the arbelos, then in Figure 2a, the radius $r_X$ is negative, the sides of triangle $XYZ$ are $a = |r_Y + r_Z|, b = |r_Z + r_X|, c = |r_X + r_Y|$, and the tripolar coordinates of $K$ relative to $XYZ$ and the area $\Delta$ of triangle $XYZ$ are

\[
KX = \lambda = |r_X + r|, \quad KY = \mu = |r_Y + r|, \quad KZ = \nu = |r_Z + r|,
\]

\[
\Delta = \sqrt{r_Xr_Yr_Z(r_X + r_Y + r_Z)}.
\]

**Figure 2a**

**Figure 2b**
If we substitute these values into the equality that the tripolar coordinates satisfy (3):
\[
(\mu^2 + \nu^2 - a^2)^2 \lambda^2 + (\nu^2 + \lambda^2 - b^2)^2 \mu^2 + (\lambda^2 + \mu^2 - c^2)^2 \nu^2
- (\mu^2 + \nu^2 - a^2)(\nu^2 + \lambda^2 - b^2)(\lambda^2 + \mu^2 - c^2) = \lambda^2 \mu^2 \nu^2,
\]
we get
\[
\left( \frac{1}{r} - \frac{1}{r_X} - \frac{1}{r_Y} - \frac{1}{r_Z} \right)^2 = \frac{4\Delta^2}{r_X r_Y r_Z}.
\]
Therefore,
\[
\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} \pm \frac{2\Delta}{r_X r_Y r_Z}.
\]

Remarks. (1) The Archimedean arbelos is of Soddy type with collinear vertices where \(r_X = R_1\), \(r_Y = R_2\), \(r_Z = -R = -(R_1 + R_2)\), \(\Delta = 0\). In this case, there is a double solution. Hence, the inradius of the Archimedean arbelos is given by
\[
\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.
\]

(2) If the radii \(r_X\), \(r_Y\), \(r_Z\) are positive, then the radii \(r\) refer to the inner and outer Soddy circles (2).

In the sequel, we shall adopt the following notation: For a line \(\ell\), \(\Pi_\ell\) denotes the orthogonal projection map onto \(\ell\).

**Lemma 3.** Let \(K(r)\) be a circle tangent externally or internally to the circles \(O_1(R_1)\) and \(O_2(R_2)\), where \(R_1, R_2\) may assume any real values. For a point \(F\) on the radical axis of the circles,
\[
r = \frac{O_1 \cdot \Pi_{O_1O_2}(FK)}{R_1 - R_2}.
\]
2.2. Let \( M \) be the midpoint of \( O_1O_2 \) (Figure 3). We have
\[
R_2^2 - R_1^2 = FO_2^2 - FO_1^2 = (FO_2 - FO_1) \cdot (FO_2 + FO_1) = 2O_1O_2 \cdot \overrightarrow{FM},
\]
\[
|R_1 + r| - |R_2 + r| = KO_2 - K_1 = (K_0 - K_1) \cdot (K_0 + K_1) = 2O_1O_2 \cdot \overrightarrow{MK}.
\]
By addition, we get
\[
2r(R_1 - R_2) = 2O_1O_2 \cdot (\overrightarrow{FM} + \overrightarrow{MK}) = 2O_1O_2 \cdot \Pi_{O_1O_2}(\overrightarrow{FK}),
\]
and the result follows.

\( \square \)

Remark. For the Archimedean twins (1) gives
\[
r_1 = \frac{(2R_1 - r_1)R_2}{R_1 + R_2 + R_1} \implies \frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.
\]

Theorem 4. Let \( U_XU_YU_Z \) be an arbelos with collinear centers \( X, Y, Z \) on a line \( L \), and \( K(r) \) be the incircle of the arbelos.

\[
r = \frac{\Pi_L(U_YU_Z)}{\frac{r_X-r_Y}{XY} - \frac{r_X-r_Z}{XZ}}.
\]

Proof. Since \( U_Z, U_Y \) are points on the radical axes of the circle pairs \( X(r_X), Y(r_Y) \), and \( X(r_X), Z(r_Z) \) respectively, by Lemma 3 we have
\[
r = \frac{XZ}{r_X-r_Y} \cdot \Pi_L(U_YU_Z) = \frac{XZ}{r_X-r_Y} \cdot \Pi_L(U_YU_Z)
\]
Since \( X, Y, Z \) are all on the line \( L \), \( \Pi_L(U_YU_Z) = \frac{r(r_X-r_Y)}{XY} \overrightarrow{j} \) and \( \Pi_L(U_YU_Z) = \frac{r(r_X-r_Z)}{XZ} \overrightarrow{j} \), where \( \overrightarrow{j} \) is a unit vector on \( L \). Hence,
\[
\Pi_L(U_YU_Z) = \Pi_L(U_YU_Z) - \Pi_L(U_YU_Z)
\]
\[
= r \left( \frac{r_X-r_Z}{XZ} - \frac{r_X-r_Y}{XY} \right) \overrightarrow{j}
\]
From this the result follows.

\( \square \)

2.2. Arbelos of type A. On a line \( L \) we take the consecutive points \( U'_Z, U_Y, U_Z, U'_Y \), and construct on the same side of the line the semicircles \( (U'_ZU_Z), (U'_YU'_Y) \), and \( (U_YU_Z) \). Let \( U_X \) be the intersection of the first two semicircles (see Figure 4). The arbelos \( U_XU_YU_Z \) is of type A. It has arc \( U_YU_Z \) positive, and \( U_ZU_X, U_XU_Y \) both negative. The diameter \( U_YU_Z \) is the base of the arbelos.

2.2.1. The incircle. Let \( K(r) \) be the incircle of this arbelos and \( A, B, C \) the points of tangency. If \( S \) is the external center of similitude of the semicircles \( (U'_ZU_Z) \) and \( (U_YU'_Y) \), then the line \( BC \) passes through \( S \), and the inversion with pole \( S \) and power \( d^2 = SU_Y \cdot SU_Z = SU_X^2 = SB \cdot SC \) swaps the semicircles \( (U'_ZU_Z) \) and \( (U_YU'_Y) \), and leaves the circle \( K(r) \) and the semicircle \( (U_YU_Z) \) invariant. Hence, \( SA^2 = SB \cdot SC \), and the \( SA \) is tangent at \( A \) to both \( K(r) \) and \( (U_YU_Z) \). If the line \( SA \) meets the perpendiculars to \( L \) at \( U_Y, U_Z \) at \( D \) and \( E \) respectively, then \( D \) is the radical center of \( K(r), (U_YU_Z), (U_YU'_Y) \), and \( E \) is the radical center of \( K(r), (U_YU_Z), (U'_ZU_Z) \). Hence, \( DC = DA \) and \( EB = EA \).
Construction of the incircle. We construct the external center of similitude $S$ of the semicircles $(U'ZU_Z)$ and $(U_YU_Y')$, and take a point $A$ on the semicircle $(U_YU_Z)$ such that $SA = SU_X$. Let the line $SA$ meet the perpendiculars to $L$ through $U_Y, U_Z$ at $D, E$ respectively. We take on $(U'ZU_Z)$ the point $B$ such that $EB = EA$, and on $(U_YU_Y')$ the point $C$ such that $DC = DA$. The circumcircle of $ABC$ is the incircle of the arbelos.

The radius of the incircle. If we take $U'ZU_Y = 2y, U_YU_Z = 2R, U_ZU_Y' = 2z$, then $r_X = R, r_Y = -R - y, r_Z = -R - z, XY = -y$, and $XZ = z$. From Theorem 4, we have

$$r = \frac{2R}{\frac{2R+y}{-y} - \frac{2R+z}{z}} = \frac{Ryz}{R(y+z)+yz},$$

or

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} + \frac{1}{z}. \quad (2)$$

2.3. Arbelos of type $B$. On a line $L$ we take the consecutive points $U'_Y, U'_Z, U_Y, U_Z$, and construct on the same side of the line the semicircles $(U'ZU_Z), (U'_YU_Y)$, and $(U_YU_Z)$. Let $U_X$ be the intersection of the first two semicircles (see Figure 5). The arbelos $U_XU_YU_Z$ is of type $B$. It has arc $U_YU_Z$ positive and the arcs $U_ZU_X, U_XU_Y$ of different signs. The base of the arbelos is $U_YU_Z$.

Construction of the incircle. The construction is the same as in type $A$, but now $S$ is the internal point of similitude of the semicircles $(U'_ZU_Z)$ and $(U'_YU_Y)$.

The radius of the incircle. We have the same formula as (2), but now since $U'_Z$ is not on the right hand side of $U_Z$, the distance $z$ must be negative, and so we have

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} - \frac{1}{z}. \quad (3)$$
Remark. For the incircle $K(r)$ of the Archimedean arbelos, since $U_X = A$, $U_Y = P$, $U_Z = B$ with base $2R_2$, $y = R_1$, $z = -R_1 - R_2$, (3) gives

$\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.$

3. Generalized Archimedean arbelos twins

We shall construct in the Archimedean arbelos a generalized pair of inscribed equal circles.

**Theorem 5.** In the Archimedean arbelos (Figure 6) where $AP = 2R_1$, $PB = 2R_2$, $AB = 2R_1 + 2R_2$, we extend $AB$ (to left and right) with equal segments $AA' = 2x = BB'$. The semicircle $(A'P)$ divides the arbelos in two arbeloi with incircles $K_1(r_1)$, $K_3(r_3)$, and the semicircle $(PB')$ divides the arbelos in two arbeloi with incircles $K_2(r_2)$, $K_4(r_4)$. We have a couple of twin circles: $r_1 = r_2$ and $r_3 = r_4$, with

$\frac{1}{r_1} = \frac{1}{x} + \frac{1}{R_1} + \frac{1}{R_2}$ and $\frac{1}{r_3} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2 + x}.$

**Proof.** The circle $K_1(r_1)$ is the incircle of an arbelos of type A with base $AP$. Hence, $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} + \frac{1}{R_2}$. 
The circle $K_2(r_2)$ is the incircle of an arbelos of type A with base $PB$. Hence, 
\[
\frac{1}{r_2} = \frac{1}{R_2} + \frac{1}{x} + \frac{1}{R_1}.
\]

The circle $K_3(r_3)$ is the incircle of an arbelos of type B with base $PB$. Hence, 
\[
\frac{1}{r_3} = \frac{1}{R_2} + \frac{1}{R_1} - \frac{1}{R_1 + R_2 + x}.
\]

The circle $K_4(r_4)$ is the incircle of an arbelos of type B with base $AP$. Hence, 
\[
\frac{1}{r_4} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2 + x}.
\]

Therefore, $r_1 = r_2$ and $r_3 = r_4$. □

**Remark.** If $x \to \infty$, then the semicircles $(A'P)$ and $(PB')$ tend to the perpendicular semiline $PQ$, and the four incircles tend to the Archimedean twin circles.

### 4. Bisectors of the Archimedean Arbelos

We have seen that in the Archimedean arbelos, the semiline $PQ$ divides the arbelos in two arbeloi with equal incircles. This semiline as a degenerate semicircle is like a bisector from $P$ of the arbelos that produces the twins (Archimedean circles) with radius $r_P$ such that
\[
\frac{1}{r_P} = \frac{1}{R_1} + \frac{1}{R_2}.
\]

We find the other two bisectors of the arbelos from the vertices $A$ and $B$ (Figure 7).

Let the semicircle $(AO_2)$ bisector of the arbelos meet the semicircle $(PB)$ at $Q_A$, and $A_1(r_1), A_2(r_2)$ be the incircles of the arbeloi $Q_AAP, Q_AAB$.

The arbelos $Q_AAP$ with base $AP$ is of type B, so we have $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} - \frac{1}{R_1 + R_2}$. In order to have $r_1 = r_2 = r_A$, we set $2x = R_2$. Hence the point $O_2$ is the midpoint of $PB$, and the bisector semicircle $(AO_2)$ meets $PQ$ at the point $Q'$ such that $PQ' = \sqrt{AP \cdot PO_2} = \sqrt{2R_1R_2}$, and $\frac{1}{r_A} = \frac{1}{R_1} + \frac{2}{R_2} - \frac{1}{R_1 + R_2}$.

Similarly, if $O_1$ is the midpoint of $AP$, then the semicircle $(O_1B)$ is the bisector of the arbelos from $B$ that passes also from $Q'$, and $\frac{1}{r_B} = \frac{2}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}$.

Hence, the three bisectors of the Archimedean Arbelos are concurrent at $Q'$. 
5. Infinitely many Archimedean circles

There are many exciting constructions of Archimedean circles or families of these; see [4, 5].

Here we construct a more natural family of Archimedean circles that contains the original Archimedean twins. In the Archimedean arbelos with diameters $AP$, $PB$, $AB$, and semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, we take at the left of $A$ the point $X$, and at right of $B$ the point $Y$ such that $AX = BY = 2x$. We rotate clockwise $P$ around $A$ by an angle $\frac{\pi}{2}$ at $A_1$, and counterclockwise $P$ around $B$ by an angle $\frac{\pi}{2}$ at $B_1$. The line $XA_1$ meets the line $PQ$ at $D_1$. The line $YB_1$ meets the line $PQ$ at $C_1$. We take at the left of $A$ the point $C$ such that $CA = PC_1 = 2d_1$, and at right of $B$ the point $D$ such that $BD = PD_1 = 2d_2$. The semicircles $(CP)$, $(AY)$ meet at $M$, and the semicircles $(PD)$, $(XB)$ meet at $N$. We show that the incircles $K_1(r_1)$, $K_2(r_2)$ of the arbeloi $MAP$, $NPB$ are both Archimedean circles.

**Proof.** Since in triangle $XPD_1$, $PA_1$ is a bisector and $AA_1$ is parallel to $PD_1$ (Figure 8), we know that $\frac{1}{PD_1} + \frac{1}{XP} = \frac{1}{AA_1}$. Hence,

$$\frac{1}{d_2} + \frac{1}{x + R_1} = \frac{1}{R_1}. \quad (4)$$

Similarly, from triangle $PYC_1$, we have

$$\frac{1}{d_1} + \frac{1}{x + R_2} = \frac{1}{R_2}. \quad (5)$$
The arbelos \( MAP \) is of type A; so we have \( r_1 = \frac{1}{R_1} + \frac{1}{d_1} + \frac{1}{x+R_2} = \frac{1}{R_1} + \frac{1}{R_2} \) from (5).

The arbelos \( NPB \) is also of type A; so we have \( r_2 = \frac{1}{R_2} + \frac{1}{d_2} + \frac{1}{x+R_1} = \frac{1}{R_2} + \frac{1}{R_1} \) from (4).

Hence \( r_1 = r_2 \) is the radius of the Archimedean circle. \( \square \)

For \( x = 0 \) the circles \( K_1(r_1), K_2(r_2) \) coincide with the original Archimedean twin circles since the semicircles \((AY), (XB)\) coincide with the semicircle \((AB)\), and the semicircles \((CP), (PD)\) coincide with the line \( PQ \). If \( M_1, M_2 \) are the midpoints of \( CP, AY \), then applying Stewart’s theorem to triangle \( MM_1M_2 \), we have

\[
O_1M^2 \cdot M_1M_2 = M_1M^2 \cdot O_1M_2 + M_2M^2 \cdot O_1M_1 - O_1M_1 \cdot O_1M_2 \cdot M_1M_2,
\]

or

\[
(d_1 + R_2 + x)O_1M^2 = (d_1 + R_1)^2(R_2 + x) + (R_1 + R_2 + x)^2d_1 - d_1(R_2 + x)(d_1 + R_2 + x).
\]

Substituting \( d_1 = \frac{R_2(x+R_2)}{x} \), we get

\[
O_1M^2 = R_1^2 + 4R_1R_2 = O_1P^2 + PQ^2 = O_1Q^2.
\]

Hence, the locus of \( M \) is the circular arc \( O_1(Q) \) from the point \( Q \) to \( Q_A \) on the perpendicular \( QA \) to \( AB \). Similarly, we can prove that the locus of \( N \) is the circular arc \( O_2(Q) \) from the point \( Q \) to \( Q_B \) on the perpendicular \( QB \) to \( AB \).

### 6. A special generalization of arbelos with arcs of angle \( 2\phi \)

If we substitute the semicircles in Archimedean arbelos with arcs of angle \( 2\phi \), i.e., \( AP = 2R_1\sin \phi \), \( PB = 2R_2\sin \phi \), \( AB = 2(R_1 + R_2)\sin \phi \) (Figure 9) [1], the points \( A, O_1, O \) are collinear; so are the points \( O, O_2, B \). The tangent to the arc \((AP)\) at \( P \) meets the arc \((AB)\) at \( Q_A \), and the tangent to the arc \((PB)\) at \( P \) meets the arc \((AB)\) at \( Q_B \). Let \( K_1(r_1), K_2(r_2) \) be the incircles of the arbeloi \( Q_BAP \) and \( Q_APB \). If \( 2\phi = \pi \), then we have the classical arbelos, and \( Q_A, Q_B \) coincide with the point \( Q \). We prove that \( r_1 = r_2 \).

**Proof:** The point \( A \) is on the radical axis of \( O_1(R_1), O_2(R_1 + R_2) \). From (1), we have

\[
r_1 = \frac{\overrightarrow{O_1O} \cdot \Pi_{O_1O}(\overrightarrow{AK_1})}{r_{O_1} - r_O} = \frac{R_2 \cdot \Pi_{O_1O}(\overrightarrow{AK_1})}{R_1 + R_1 + R_2}.
\]

Also, \( r_1 = \Pi_{O_1O}(K_1P) \). Hence,

\[
2R_1\sin^2 \phi = \Pi_{O_1O}(AK_1) + \Pi_{O_1O}(K_1P) = r_1 \cdot \frac{2R_1 + R_2}{R_2} + r_1 = r_1 \cdot \frac{2(R_1 + R_2)}{R_2},
\]

or \( r_1 = \frac{R_1R_2\sin^2 \phi}{R_1 + R_2} \). Similarly,

\[
r_2 = \frac{\overrightarrow{O_2O} \cdot \Pi_{O_2O}(\overrightarrow{BK_2})}{r_O - r_{O_2}} = \frac{R_1 \cdot \Pi_{O_2O}(BK_2)}{-R_1 - R_2 - R_2},
\]

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and \( r_2 = \Pi_{OO_2}(PK_2) \). Hence,

\[
2R_2 \sin^2 \phi = \Pi_{OO_2}(PK_2) + \Pi_{OO_2}(K_2B) = r_2 + r_2 \cdot \frac{R_1 + 2R_2}{R_1} = r_2 \cdot \frac{2(R_1 + R_2)}{R_1},
\]

or \( r_2 = \frac{R_1 R_2 \sin^2 \phi}{R_1 + R_2} \).

If \( 2\phi = \pi \), then we have the radius of the Archimedean circle. □

Construction of the twins. The perpendicular from \( P \) to \( AB \) meets \( O_1O_2 \) at the point \( C \), and the parallel from \( C \) to \( PO_1 \) meets \( PO_2 \) at \( D \). Since \( PC \) is a bisector in triangle \( PO_1O_2 \), we have \( \frac{1}{CD} = \frac{1}{R_1} + \frac{1}{R_2} \). Hence we need the construction of \( r_1 = CD \cdot \sin^2 \phi \). The perpendicular from \( D \) to \( AB \) meets \( AB \) at \( E \) and the perpendicular from \( E \) to \( PO_1 \) meets this line at the point \( F_1 \). The symmetric of \( F_1 \) in \( PC \) is the point \( F_2 \) on the line \( PO_2 \). The perpendicular at \( F_2 \) to \( PF_2 \) meets the circle \( O_1(F_1) \) at the point \( K_1 \) and the line \( EF_1 \) meets the circle \( O_2(F_2) \) at the point \( K_2 \). These points are the centers of the twin incircles and the construction of these circles is obvious.

References


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