Problem A1 Let \( n \) be a positive integer. How many ways are there to write \( n \) as a sum of positive integers,

\[
n = a_1 + a_2 + \cdots + a_k,
\]

with \( k \) an arbitrary positive integer and \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1} \)?

Solution There are \( n \) ways. It suffices to show that if \( k \) is an integer, \( 1 \leq k \leq n \), there is exactly one way to write \( n \) as a sum of \( k \) integers of the specified type. By the quotient algorithm, \( n = kq + r \) for some \( r \), \( 0 \leq r \leq k - 1 \). If \( r = 0 \), we can take \( a_1 = a_2 = \cdots = a_k = q \) and write \( n = a_1 + \cdots + a_k \). If \( 0 < r \leq k - 1 \), we can take \( a_1 = \cdots = a_{k-r} = q, a_{k-r+1} = \cdots = a_k = q + 1 \) to get \( a_1 + \cdots + a_k = (k-r)q + r(q+1) = n \). This shows there is at least one way to decompose \( n \) as specified for each \( k \). To see this is the only such way, notice that if \( n = a_1 + a_2 + \cdots + a_k \) with \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1} \leq a_1 + 1 \), then we either have \( a_1 = a_2 = \cdots = a_k \) or there is \( \ell, 1 < \ell < k \) such that \( a_1 = a_2 = \cdots = a_{\ell} < a_{\ell+1} = \cdots = a_k = a_1 + 1 \). In the former case \( n = k a_1 \), so that we are in the case \( r = 0, q = a_1 \), and the decomposition is the one given for that case. In the latter,

\[
n = \ell a_1 + (k - \ell)(a_1 + 1) = k a_1 + (\ell - k)
\]

so that we are in the case \( r = k - \ell > 0 \), and the decomposition coincides with the previous one. This proves the uniqueness of the decomposition for each \( k \).