**Problem A4** Suppose that \(a, b, c, A, B, C\) are real numbers, \(a \neq 0, A \neq 0\), such that
\[
|ax^2 + bx + c| \leq |Ax^2 + Bx^2 + C|
\]
for all real numbers \(x\). Show that
\[
|b^2 - 4ac| \leq |B^2 - 4AC|.
\]

**Solution** Dividing by \(x\) and letting \(x \to \infty\), we see that \(|a| \leq |A|\).

We consider three cases. **Case 1.** \(B^2 - 4AC > 0\). Then the equation \(Ax^2 + Bx + C = 0\) has two distinct roots. It follows that these must also be roots of the equation \(ax^2 + bx + c = 0\), and the differences of the two roots of the first equation must coincide with the difference of the roots of the second. That is,
\[
\frac{\sqrt{B^2 - 4AC}}{A} = \frac{\sqrt{b^2 - 4ac}}{a},
\]

hence
\[
\sqrt{B^2 - 4AC} = \frac{A}{a} \sqrt{b^2 - 4ac} \geq \sqrt{b^2 - 4ac},
\]

and the desired inequality follows.

**Case 2.** \(B^2 - 4AC = 0\) so that \(Ax^2 + Bx + C = 0\) has a double root at \(-B/2A\). The assumed inequality implies that
\[
\limsup_{x \to A} \frac{|ax^2 + bx + c|}{(x + \frac{b}{2A})^2} < \infty,
\]

implying that \(ax^2 + bx + c = 0\) also has a double root at \(-B/2A\). Thus \(-B/2A = -b/2a\) and in this case we have \(b^2 - 4ac = 0 = B^2 - 4AC\).

**Case 3.** \(B^2 - 4AC < 0\). The equation \(Ax^2 + Bx + C = 0\) has no real roots. Assume \(A > 0\), the case \(A < 0\) is similar. In this case it is geometrically clear (and easy to prove analytically) that the \(y\)-coordinate of the vertex of the parabola \(y = Ax^2 + Bx + C\) must be larger than the \(y\) coordinates of the vertices of the parabolas of equations \(y = \pm(ax^2 + bx + c)\); i.e.,
\[
\frac{4AC - B^2}{4A} \geq \frac{|4ac - b^2|}{4|a|},
\]

hence
\[
4AC - B^2 \geq \frac{A}{|a|} |4ac - b^2| \geq |4ac - b^2|.
\]

The inequality has been proved in all cases.