**Problem B6** Let \( f(x) \) be a continuous real-valued function defined on the interval \([0, 1]\). Show that

\[
\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.
\]

**Solution** Let \( A = \{x \in [0, 1] : f(x) \geq 0\} \), \( B = \{x \in [0, 1] : f(x) < 0\} \) and let \(|A|, |B|\) be the Lebesgue measures of \( A, B \) respectively. Then \( 0 \leq |A|, |B|, |A| + |B| = 1 \). Setting

\[
\mu = \int_A f(x) \, dx = \int_A |f(x)| \, dx,
\]

\[
\nu = -\int_B f(x) \, dx = \int_B |f(x)| \, dx,
\]

we get at once

\[
\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy = \int_{A \times A} (f(x) + f(y)) \, dx \, dy
\]

\[
+ 2 \int_{A \times B} |f(x) - f(y)| \, dx \, dy - \int_{B \times B} (f(x) + f(y)) \, dx \, dy
\]

\[
\geq \int_{A \times A} (f(x) + f(y)) \, dx \, dy + 2 \left| \int_{A \times B} (f(x) - f(y)) \, dx \, dy \right|
\]

\[
- \int_{B \times B} (f(x) + f(y)) \, dx \, dy = 2|A|\mu + 2|B|\nu + 2 |A|\nu - |B|\mu.
\]

We proved the following two inequalities

\[
\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq 2(|A| - |B|)\mu + 2\nu,
\]

\[
\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq 2\mu + 2(|B| - |A|)\nu.
\]

Now \(-1 \leq \gamma := |A| - |B| \leq 1\). If \( \gamma \geq 0 \), set \( t = (1 + 2\gamma)/(2 + 2\gamma) \in (0, 1) \). Multiplying the first of the two inequalities just mentioned by \( t \), the second one by \( 1 - t \) and adding yields

\[
\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq 2\gamma t\mu + 2t\nu + (1 - t)\mu - 2(1 - t)\gamma
\]

\[
= \frac{2 + 2\gamma + 4\gamma^2}{2 + 2\gamma} \mu + \nu \geq \mu + \nu = \int_0^1 |f(x)| \, dx.
\]

If \( \gamma < 0 \), we achieve the same result setting \( t = (1 - 2\gamma)/(2 - 2\gamma) \), multiplying the first inequality by \( 1 - t \), the second one by \( t \), and adding.