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The Isogonal Tripolar Conic

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Abstract. In trilinear coordinates with respect to a given triangle ABC, we define the isogonal tripolar of a point $P(p,q,r)$ to be the line p: $p\alpha+q\beta+r\gamma=0$. We construct a unique conic Φ, called the isogonal tripolar conic, with respect to which p is the polar of P for all P. Although the conic is imaginary, it has a real center and real axes coinciding with the center and axes of the real orthic inconic. Since ABC is self-conjugate with respect to Φ, the imaginary conic is harmonically related to every circumconic and inconic of ABC. In particular, Φ is the reciprocal conic of the circumcircle and Steiner's inscribed ellipse. We also construct an analogous isotomic tripolar conic Ψ by working with barycentric coordinates.

1. Trilinear coordinates

For any point P in the plane ABC, we can locate the right projections of P on the sides of triangle ABC at P_1, P_2, P_3 and measure the distances PP_1, PP_2 and PP_3. If the distances are directed, i.e., measured positively in the direction of each vertex to the opposite side, we can identify the distances $\underline{\alpha}=\overrightarrow{PP_1}$, $\underline{\beta}=\overrightarrow{PP_2}$, $\underline{\gamma}=\overrightarrow{PP_3}$ (Figure 1) such that

$$a\underline{\alpha}+b\underline{\beta}+c\underline{\gamma}=2\triangle$$

where a, b, c, $\triangle$ are the side lengths and area of triangle ABC. This areal equation for all positions of P means that the ratio of the distances is sufficient to define the *trilinear coordinates* of $P(\alpha,\beta,\gamma)$ where

$$\alpha:\beta:\gamma=\underline{\alpha}:\underline{\beta}:\underline{\gamma}.$$

For example, if we consider the coordinates of the vertex A, the incenter I, and the first excenter I_1, we have absolute $\underline{\alpha\beta\gamma}-$coordinates : $A(h_1,0,0)$, $I(r,r,r)$, $I_1(-r_1,r_1,r_1)$, where h_1, r, r_1 are respectively the altitude from A, the inradius and the first exradius of triangle ABC. It follows that the trilinear $\alpha\beta\gamma-$coordinates in their simplest form are $A(1,0,0)$, $I(1,1,1)$, $I_1(-1,1,1)$. Let R be the circumradius, and h_1, h_2, h_3 the altitudes, so that $ah_1=bh_2=ch_3=2\triangle$. The absolute coordinates of the circumcenter O, the orthocenter H, and the median point [1] G are $O(R\cos A,\ R\cos B,\ R\cos C)$, $H(2R\cos B\cos C,\ 2R\cos C\cos A,$

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[1]The median point is also known as the centroid.

$2R\cos A\cos B$), and $G(\frac{h_1}{3}, \frac{h_2}{3}, \frac{h_3}{3})$, giving trilinear coordinates: $O(\cos A$, $\cos B$, $\cos C)$, $H(\sec A$, $\sec B$, $\sec C)$, and $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$.

2. Isogonal conjugate

For any position of P we can define its isogonal conjugate Q such that the directed angles $(AC, AQ) = (AP, AB) = \theta_1$, $(BA, BP) = (BQ, BC) = \theta_2$, $(CB, CP) = (CQ, CA) = \theta_3$ as shown in Figure 1. If the absolute coordinates of Q are $\underline{\alpha}' = \overrightarrow{QQ_1}$, $\underline{\beta}' = \overrightarrow{QQ_2}$, $\underline{\gamma}' = \overrightarrow{QQ_3}$, then

$$\frac{PP_2}{PP_3} = \frac{AP\sin(A-\theta_1)}{AP\sin\theta_1} \quad \text{and} \quad \frac{QQ_2}{QQ_3} = \frac{AQ\sin\theta_1}{AQ\sin(A-\theta_1)}$$

so that that $PP_2 \cdot QQ_2 = PP_3 \cdot QQ_3$, implying $\underline{\beta\beta'} = \underline{\gamma\gamma'}$. Similarly, $\underline{\alpha\alpha'} = \underline{\beta\beta'}$ and $\underline{\gamma\gamma'} = \underline{\alpha\alpha'}$, so that $\underline{\alpha\alpha'} = \underline{\beta\beta'} = \underline{\gamma\gamma'}$. Consequently, $\alpha\alpha' = \beta\beta' = \gamma\gamma'$.

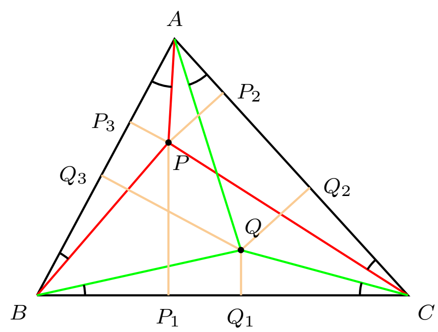

Figure 1

Hence, Q is the triangular inverse of P; i.e., if P has coordinates (α, β, γ), then its isogonal conjugate Q has coordinates $(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma})$. It will be convenient to use the notation $\hat{P}$ for the isogonal conjugate of P. We can immediately note that $O(\cos A, \cos B, \cos C)$ and $H(\sec A, \sec B, \sec C)$ are isogonal conjugates. On the other hand, the symmedian point K, being the isogonal conjugate of $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$, has coordinates $K(a, b, c)$, i.e., the distances from K to the sides of triangle ABC are proportional to the side lengths of ABC.

3. Tripolar

We can now define the *line coordinates* (l, m, n) of a given line ℓ in the plane ABC, such that any point P with coordinates (α, β, γ) lying on ℓ must satisfy the linear equation $l\alpha + m\beta + n\gamma = 0$. In particular, the side lines BC, CA, AB have line coordinates (1,0,0), (0,1,0), (0,0,1), with equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$ respectively.

A specific line that may be defined is the harmonic or trilinear polar of Q with respect to ABC, which will be called the *tripolar* of Q.

In Figure 2, $L'M'N'$ is the tripolar of Q, where LMN is the diagonal triangle of the quadrangle $ABCQ$; and $L'M'N'$ is the axis of perspective of the triangles ABC and LMN. Any line through Q meeting two sides of ABC at U, V and meeting $L'M'N'$ at W creates an harmonic range $(UV;QW)$. To find the line coordinates of $L'M'N'$ when Q has coordinates (p',q',r'), we note $L = AQ \cap BC$ has coordinates $(0,q',r')$, since $\frac{LL_2}{LL_3} = \frac{QQ_2}{QQ_3}$. Similarly for $M(p',0,r')$ and $N(p',q',0)$. Hence the equation of the line MN is

$$\frac{\alpha}{p'} = \frac{\beta}{q'} + \frac{\gamma}{r'} \tag{1}$$

since the equation is satisfied when the coordinates of M or N are substituted for α, β, γ in (1). So the coordinates of $L' = MN \cap BC$ are $L'(0,q',-r')$. Similarly for $M'(p',0,-r')$ and $N'(p',-q',0)$, leading to the equation of the line $L'M'N'$:

$$\frac{\alpha}{p'} + \frac{\beta}{q'} + \frac{\gamma}{r'} = 0. \tag{2}$$

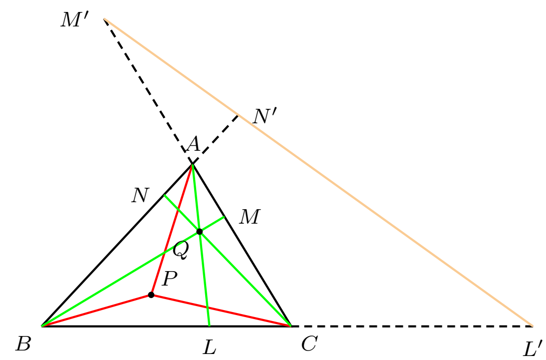

Figure 2

Now from the previous analysis, if $P(p,q,r)$ and $Q(p',q',r')$ are isogonal conjugates then $pp' = qq' = rr'$ so that from (2) the equation of the line $L'M'N'$ is $p\alpha + q\beta + r\gamma = 0$. In other words, the line coordinates of the tripolar of Q are the trilinear coordinates of P. We can then define the *isogonal tripolar* of $P(p,q,r)$ as the line $L'M'N'$ with equation $p\alpha + q\beta + r\gamma = 0$.

For example, for the vertices $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, the isogonal tripolars are the corresponding sides BC ($\alpha = 0$), CA ($\beta = 0$), AB ($\gamma = 0$). For the notable points $O(\cos A, \cos B, \cos C)$, $I(1,1,1)$, $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$, and $K(a,b,c)$, the corresponding isogonal tripolars are

$$\begin{aligned}
\mathsf{o}: &\quad \alpha\cos A + \beta\cos B + \gamma\cos C = 0,\\
\mathsf{i}: &\quad \alpha + \beta + \gamma = 0,\\
\mathsf{g}: &\quad \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0,\\
\mathsf{k}: &\quad a\alpha + b\beta + c\gamma = 0.
\end{aligned}$$

Here, o, i, g, k are respectively the orthic axis, the anti-orthic axis, Lemoine's line, and the line at infinity, i.e., the tripolars of H, I, K, and G. Clark Kimberling has assembled a catalogue of notable points and notable lines with their coordinates in a contemporary publication [3].

4. The isogonal tripolar conic Φ

Now consider a point $P_2(p_2, q_2, r_2)$ on the isogonal tripolar of $P_1(p_1, q_1, r_1)$, i.e., the line

$$\mathsf{p}_1 : \qquad p_1\alpha + q_1\beta + r_1\gamma = 0.$$

Obviously P_1 lies on the isogonal tripolar of P_2 since the equality $p_1p_2 + q_1q_2 + r_1r_2 = 0$ is the condition for both incidences. Furthermore, the line P_1P_2 has equation

$$(q_1r_2 - q_2r_1)\alpha + (r_1p_2 - r_2p_1)\beta + (p_1q_2 - p_2q_1)\gamma = 0,$$

while the point $\mathsf{p}_1 \cap \mathsf{p}_2$ has coordinates $(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1)$. It follows that $\mathsf{t} = P_1P_2$ is the isogonal tripolar of $T = \mathsf{p}_1 \cap \mathsf{p}_2$. These isogonal tripolars immediately suggest the classical polar reciprocal relationships of a geometrical conic. In fact, the triangle P_1P_2T has the analogous properties of a self-conjugate triangle with respect to a conic, since each side of triangle P_1P_2T is the isogonal tripolar of the opposite vertex. This means that a significant conic could be drawn self-polar to triangle P_1P_2T. But an infinite number of conics can be drawn self-polar to a given triangle; and a further point with its polar are required to identify a unique conic [5]. We can select an arbitrary point P_3 with its isogonal tripolar p_3 for this purpose. Now the equation to the general conic in trilinear coordinates is [4]

$$\mathcal{S} : \qquad l\alpha^2 + m\beta^2 + n\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

and the polar of $P_1(p_1, q_1, r_1)$ with respect to $\mathcal{S}$ is

$$\mathsf{s}_1 : \quad (lp_1 + hq_1 + gr_1)\alpha + (hp_1 + mq_1 + fr_1)\beta + (gp_1 + fq_1 + nr_1)\gamma = 0.$$

By definition we propose that for $i = 1, 2, 3$, the lines p_i and s_i coincide, so that the line coordinates of p_i and s_i must be proportional; i.e.,

$$\frac{lp_i + hq_i + gr_i}{p_i} = \frac{hp_i + mq_i + fr_i}{q_i} = \frac{gp_i + fq_i + nr_i}{r_i}.$$

Solving these three sets of simultaneous equations, after some manipulation we find that $l = m = n$ and $f = g = h = 0$, so that the equation of the required conic is $\alpha^2 + \beta^2 + \gamma^2 = 0$. This we designate the *isogonal tripolar conic* Φ.

From the analysis Φ is the unique conic which reciprocates the points P_1, P_2, P_3 to the lines p_1, p_2, p_3. But any set of points P_i, P_j, P_k with the corresponding isogonal tripolars p_i, p_j, p_k could have been chosen, leading to the same equation for the reciprocal conic. We conclude that *the isogonal tripolar of any point P in the plane ABC is the polar of P with respect to* Φ. Any triangle $P_iP_jT_k$ with $T_k = \mathsf{p}_i \cap \mathsf{p}_j$ is self-conjugate with respect to Φ. In particular, the basic triangle ABC is self-conjugate with respect to Φ, since each side is the isogonal tripolar of its opposite vertex.

From the form of the equation $\alpha^2 + \beta^2 + \gamma^2 = 0$, the isogonal tripolar conic Φ is obviously an imaginary conic. So the conic exists on the complex projective plane. However, it will be shown that the imaginary conic has a real center and real axes; and that Φ is the reciprocal conic of a pair of notable real conics.

5. The center of Φ

To find the center of Φ, we recall that the polar of the center of a conic with respect to that conic is the line at infinty ℓ_∞ which we have already identified as $\mathsf{k}:\ a\alpha + b\beta + c\gamma = 0$, the isogonal tripolar of the symmedian point $K(a,b,c)$. So the center of Φ and the center of its director circle are situated at K. From Gaskin's Theorem, the director circle of a conic is orthogonal to the circumcircle of every self-conjugate triangle. Choosing the basic triangle ABC as the self-conjugate triangle with circumcenter O and circumradius R, we have $\rho^2 + R^2 = OK^2$, where ρ is the director radius of Φ. But it is known [2] that $R^2 - OK^2 = 3\mu^2$, where $\mu = \dfrac{abc}{a^2+b^2+c^2}$ is the radius of the cosine circle of ABC. From this,

$$\rho = i\sqrt{3}\mu = i\sqrt{3}\cdot\frac{abc}{a^2+b^2+c^2}.$$

6. Some lemmas

To locate the axes of Φ, some preliminary results are required which can be found in the literature [1] or obtained by analysis.

Lemma 1. *If a diameter of the circumcircle of ABC meets the circumcircle at X, Y, then the isogonal conjugates of X and Y (designated $\hat{X}$, $\hat{Y}$) lie on the line at infinity; and for arbitrary P, the line $P\hat{X}$ and $P\hat{Y}$ are perpendicular.*

Here is a special case.

Lemma 2. *If the chosen diameter is the Euler line OGH, then $\hat{X}\hat{Y}$ lie on the asymptotes of Jerabek's hyperbola $\mathcal{J}$, which is the locus of the isogonal conjugate of a variable point on the Euler line OGH (Figure 3).*

Lemma 3. *If the axes of a conic $\mathcal{S}$ with center Q meets ℓ_∞ at E, F, then the polars of E, F with respect to $\mathcal{S}$ are the perpendicular lines QF, QE; and E, F are the only points on ℓ_∞ with this property.*

Lemma 4. *If UGV is a chord of the circumcircle Γ through G meeting Γ at U, V, then the tripolar of U is the line $K\hat{V}$ passing through the symmedian point K and the isogonal conjugate of V.*

7. The axes of Φ

To proceed with the location of the axes of Φ, we start with the conditions of Lemma 2 where X, Y are the common points of OGH and Γ.

From Lemma 4, since XGY are collinear, the tripolars of X, Y are respectively $K\hat{Y}$, $K\hat{X}$, which are perpendicular from Lemma 1. Now from earlier definitions, the tripolars of X, Y are the isogonal tripolars of $\hat{X}$, $\hat{Y}$, so that the isogonal tripolars of $\hat{X}$, $\hat{Y}$ are the perpendiculars $K\hat{Y}$, $K\hat{X}$ through the center of Φ. Since $\hat{X}\hat{Y}$ lie on ℓ_∞, $K\hat{X}$, $K\hat{Y}$ must be the axes of Φ from Lemma 3. And these axes are parallel to the asymptotes of $\mathcal{J}$ from Lemma 2.

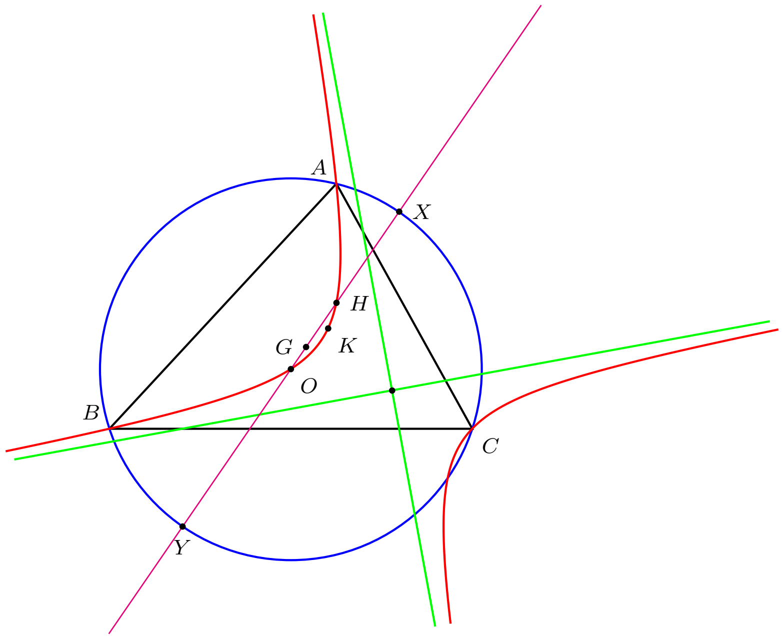

Figure 3. The Jerabek hyperbola

Now it is known [1] that the asymptotes of $\mathcal{J}$ are parallel to the axes of the orthic inconic (Figure 4). The orthic triangle has its vertices at H_1, H_2, H_3 the feet of the altitudes AH, BH, CH. The orthic inconic has its center at K and touches the sides of triangle ABC at the vertices of the orthic triangle. So the axes of the imaginary conic Φ coincide with the axes of the real orthic inconic.

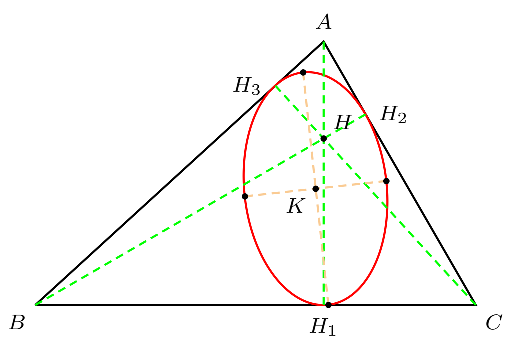

Figure 4. The orthic inconic

8. Φ as a reciprocal conic of two real conics

Although the conic Φ is imaginary, every real point P has a polar p with respect to Φ. In particular if P lies on the circumcircle Γ, its polar p touches Steiner's inscribed ellipse σ with center G. This tangency arises from the known theorem

[1] that the tripolar of any point on ℓ_∞ touches σ. From Lemma 1 this tripolar is the isogonal tripolar of the corresponding point of Γ. Now the basic triangle ABC (which is self-conjugate with respect to Φ) is inscribed in Γ and tangent to σ, which touches the sides of ABC at their midpoints (Figure 5). In the language of classical geometrical conics, the isogonal tripolar conic Φ is harmonically inscribed to Γ and harmoncially circumscribed to σ. From the tangency described above, Φ is the reciprocal conic to $\Gamma \rightleftharpoons \sigma$. Furthermore, since ABC is self-conjugate with respect to Φ, an infinite number of triangles $P_iP_jP_k$ can be drawn with its vertices inscribed in Γ, its sides touching σ, and self-conjugate with respect to Φ. Since Φ is the reciprocal conic of $\Gamma \rightleftharpoons \sigma$, for any point on σ, its polar with respect to Φ (i.e., its isogonal tripolar) touches Γ. In particular, if the tangent p_i touches σ at $T_i(u_i, v_i, w_i)$ for $i = 1, 2, 3$, then t_i, the isogonal tripolar of T_i, touches Γ at P_i (Figure 5).

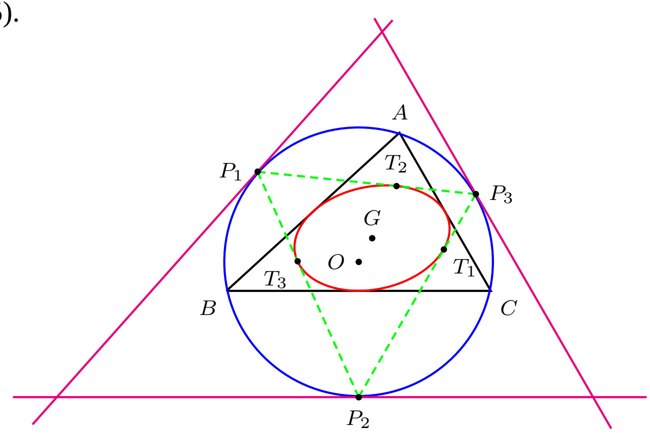

Figure 5

Now, the equation to the circumcircle Γ is $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$. The equation of the tangent to Γ at $P_i(p_i, q_i, r_i)$ is

$$(cq_i + br_i)\alpha + (ar_i + cp_i)\beta + (bp_i + aq_i)\gamma = 0.$$

If this tangent coincides with t_i, the isogonal tripolar of T_i, then the coordinates of T_i are

$$u_i = cq_i + br_i, \qquad v_i = ar_i + cp_i, \qquad w_i = bp_i + aq_i. \tag{3}$$

So, if t_i is the tangent at $P_i(p_i, q_i, r_i)$ to Γ, and simultaneously the isogonal tripolar of T_i, then the coordinates of T_i are as shown in (3). But this relationship can be generalized for any P_i in the plane of ABC, since the equation to the polar of P_i with respect to Γ is identical to the equation to the tangent at P_i (in the particular case that P_i lies on Γ). In other words, the isogonal tripolar of $T_i(u_i, v_i, w_i)$ with the coordinates shown at (3) is the polar of $P_i(p_i, q_i, r_i)$ with respect to Γ, for any P_i, T_i in the plane of ABC.

9. The isotomic tripolar conic Ψ

To find an alternative description of the transformation $P \mapsto T$, we define the *isotomic conjugate* and the *isotomic tripolar*.

In the foregoing discussion we have used trilinear coordinates (α, β, γ) to define the point P and its isogonal tripolar p. However, we could just as well use *barycentric* (areal) coordinates (x, y, z) to define P. With $\underline{x}$ = area (PBC), $\underline{y}$ = area (PCA), $\underline{z}$ = area (PAB), and $\underline{x} + \underline{y} + \underline{z}$ = area (ABC), comparing with trilinear coordinates of P we have

$$a\underline{\alpha} = 2\underline{x}, \qquad b\underline{\beta} = 2\underline{y}, \qquad c\underline{\gamma} = 2\underline{z}.$$

Using directed areas, i.e., positive area (PBC) when the perpendicular distance PP_1 is positive, the ratio of the areas is sufficient to define the (x, y, z) coordinates of P, with $x : \underline{x} = y : \underline{y} = z : \underline{z}$. The absolute coordinates $(\underline{x}, \underline{y}, \underline{z})$ can then be found from the areal coordinates (x, y, z) using the areal identity $\underline{x} + \underline{y} + \underline{z} = \triangle$. For example, the barycentric coordinates of A, I, I_1, O, H, G, K are $A(1, 0, 0)$, $I(a, b, c)$, $I_1(-a, b, c)$, $O(a\cos A, b\cos B, c\cos C)$, $H(a\sec A, b\sec B, c\sec C)$, $G(1, 1, 1)$, $K(a^2, b^2, c^2)$ respectively.

In this barycentric system we can identity the coordinates (x', y', z') of the isotomic conjugate $\overline{P}$ of P as shown in Figure 6, where $\overrightarrow{BL}=\overrightarrow{L'C}$, $\overrightarrow{CM}=\overrightarrow{M'A}$, $\overrightarrow{AN}=\overrightarrow{N'B}$. We find by the same procedure that $xx' = yy' = zz'$ for P, $\overline{P}$, so that the areal coordinates of $\overline{P}$ are $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$, explaining the alternative description that $\overline{P}$ is the triangular reciprocal of P.

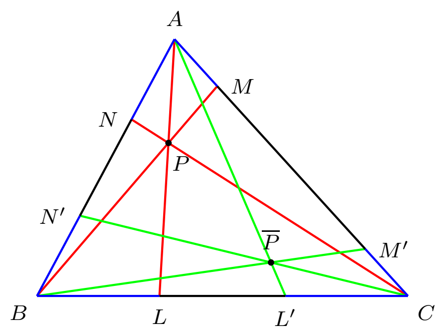

Figure 6

Following the same argument as heretofore, we can define the *isotomic tripolar* of $P(p, q, r)$ as the tripolar of $\overline{P}$ with barycentric equation $px + qy + rz = 0$, and then identify the imaginary *isotomic tripolar conic* Ψ with equation $x^2 + y^2 + z^2 = 0$. The center of Ψ is the median point $G(1, 1, 1)$ since the isotomic tripolar of G is the ℓ_∞ with barycentric equation $x + y + z = 0$. By analogous procedure we can find the axes of Ψ which coincide with the real axes of Steiner's inscribed ellipse σ.

Again, we find that the basic triangle ABC is self conjugate with respect to Ψ, and from Gaskin's Theorem, the radius of the imaginary director circle d is given by $d^2 + R^2 = OG^2$. From this, $d^2 = OG^2 - R^2 = -\frac{1}{9}(a^2 + b^2 + c^2)$, giving

$$d = \frac{i}{3}\sqrt{a^2 + b^2 + c^2}.$$

In the analogous case to Figure 5, we find that in Figure 7, if P is a variable point on Steiner's circum-ellipse θ (with center G), then the isotomic tripolar of P is tangent to σ, and Ψ is the reciprocal conic of $\theta \rightleftharpoons \sigma$. Generalizing this relationship as before, we find that the polar of P (pqr) with respect to θ is the isotomic tripolar of T with barycentric coordinates $(q+r, r+p, p+q)$. Furthermore, we can describe the transformation $P \mapsto T$ in vector geometry as $\overrightarrow{PG} = 2\ \overrightarrow{GT}$, or more succinctly that T is the complement of P [2]. The inverse transformation $T \mapsto P$ is given by $\overrightarrow{TG} = \frac{1}{2}\ \overrightarrow{GP}$, where P is the anticomplement of T. So the transformation of point T to the isotomic tripolar t can be described as

$$\begin{aligned} \text{t} &= \text{isotomic tripole of } T \\ &= \text{polar of } T \text{ with respect to } \Psi \\ &= \text{polar of } P \text{ with respect to } \theta, \end{aligned}$$

where $\overrightarrow{PG} = 2\ \overrightarrow{GT}$. In other words, the transformation of a point $P(p, q, r)$ to its isotomic tripolar $px + qy + rz = 0$ is a dilatation $(G, -2)$ followed by polar reciprocation in θ, Steiner's circum-ellipse.

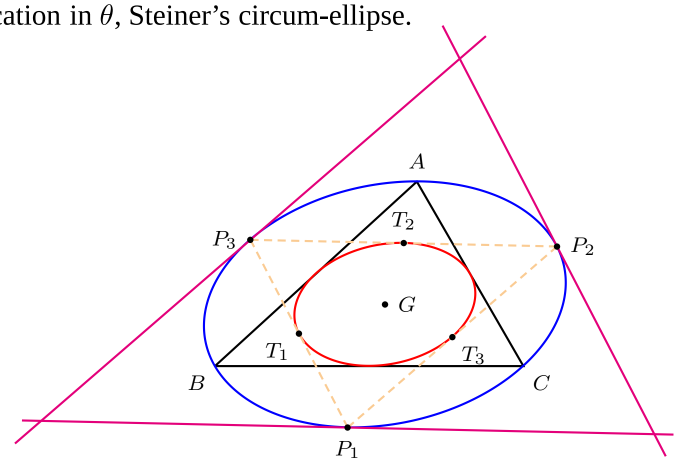

Figure 7

To find the corresponding transformation of a point to its isogonal tripolar, we recall that the polar of $P(p, q, r)$ with respect to Γ is the isogonal tripolar of T, where T has trilinear coordinates $(cq + br, ar + cp, bp + aq)$ from (3). Now, $\overline{\hat{P}}$, the isotomic conjugate of the isogonal conjugate of P, has coordinates $(\frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2})$ [3]. Putting $R = \overline{\hat{P}}$, the complement of R has coordinates $(cq + br, ar + cp, bp + aq)$, which are the coordinates of T. So the transformation of point T to its isogonal tripolar t can be described as

$$\begin{aligned} \text{t} &= \text{isogonal tripolar of } T \\ &= \text{polar of } T \text{ with respect to } \Phi \\ &= \text{polar of } P \text{ with respect to } \Gamma, \end{aligned}$$

where $\overrightarrow{RG} = 2\ \overrightarrow{GT}$, and $P = \hat{\overline{R}}$, the isogonal conjugate of the isotomic conjugate of R. In other words, the transformation of a point P with trilinear coordinates

(p, q, r) to its isogonal tripolar $(p\alpha + q\beta + r\gamma = 0)$ is a dilatation $(G, -2)$, followed by isotomic transformation, then isogonal transformation, and finally polar reciprocation in the circumcircle Γ.

We conclude with the remark that the two well known systems of homogeneous coordinates, viz. trilinear (α, β, γ) and barycentric (x, y, z), generate two analogous imaginary conics Φ and Ψ, whose real centers and real axes coincide with the corresponding elements of notable real inconics of the triangle. In each case, the imaginary conic reciprocates an arbitrary point P to the corresponding line p, whose line coordinates are identical to the point coordinates of P. And in each case, reciprocation in the imaginary conic is the equivalent of well known transformations of the real plane.

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