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The Malfatti Problem

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Abstract. A solution is given of Steiner's variation of the classical Malfatti problem in which the triangle is replaced by three circles mutually tangent to each other externally. The two circles tangent to the three given ones, presently known as Soddy's circles, are encountered as well.

In this well known problem, construction is sought for three circles C_1', C_2' and C_3', tangent to each other pairwise, and of which C_1' is tangent to the sides A_1A_2 and A_1A_3 of a given triangle $A_1A_2A_3$, while C_2' is tangent to A_2A_3 and A_2A_1 and C_3' to A_3A_1 and A_3A_2. The problem was posed by Malfatti in 1803 and solved by him with the help of an algebraic analysis. Very well known is the extraordinarily elegant geometric solution that Steiner announced without proof in 1826. This solution, together with the proof Hart gave in 1857, one can find in various textbooks.[1] Steiner has also considered extensions of the problem and given solutions. The first is the one where the lines A_2A_3, A_3A_1 and A_1A_2 are replaced by circles. Further generalizations concern the figures of three circles on a sphere, and of three conic sections on a quadric surface. In the nineteenth century many mathematicians have worked on this problem. Among these were Cayley (1852) [2], Schellbach (who in 1853 published a very nice goniometric solution), and Clebsch (who in 1857 extended Schellbach's solution to three conic sections on a quadric surface, and for that he made use of elliptic functions). If one allows in Malfatti's original problem also escribed and internally tangent circles, then there are a total of 32 (real) solutions. One can find all these solutions mentioned by Pampuch (1904).[3] The generalizations mentioned above even have, as appears from investigation by Clebsch, 64 solutions.

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The present article is one, *Verscheidenheid* XXVI , in a series by Oene Bottema (1901-1992) in the periodical *Euclides* of the Dutch Association of Mathematics Teachers. A collection of articles from this series was published in 1978 in form of a book [1]. The original article does not contain any footnote nor bibliography. All annotations, unless otherwise specified, are by the translator. Some illustrative diagrams are added in the Appendix.

[1] See, for examples, [3, 5, 7, 8, 9].

[2] Cayley's paper [4] was published in 1854.

[3] Pampuch [11, 12].

The literature about the problem is so vast and widespread that it is hardly possible to consult completely. As far as we have been able to check, the following special case of the generalization by Steiner has not drawn attention. It is attractive by the simplicity of the results and by the possibility of a certain stereometric interpretation.

The problem of Malfatti-Steiner is as follows: Given are three circles C_1, C_2 and C_3. Three circles C_1', C_2' and C_3' are sought such that C_1' is tangent to C_2, C_3, C_2' and C_3', the circle C_2' to C_3, C_1, C_3' and C_1', and, C_3' to C_1, C_2, C_1' and C_2'. Now we examine the special case, where the *three given circles* C_1, C_2, C_3 *are pairwise tangent as well*.

This problem certainly can be solved following Steiner's general method. We choose another route, in which the simplicity of the problem appears immediately. If one applies an *inversion* with center the point of tangency of C_2 and C_3, then these two circles are transformed into two parallel lines ℓ_2 and ℓ_3, and C_1 into a circle K tangent to both (Figure 1). In this figure the construction of the required circles K_i is very simple. If the distance between ℓ_2 and ℓ_3 is $4r$, then the radii of K_2 and K_3 are equal to r, that of K_1 equal to $2r$, while the distance of the centers of K and K_1 is equal to $4r\sqrt{2}$. Clearly, the problem always has *two* (real) solutions.[4]

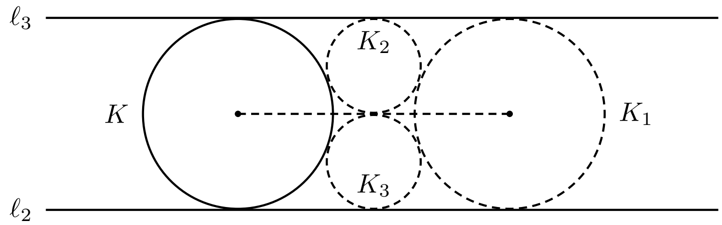

Figure 1

Our goal is the computation of the radii R_1', R_2' and R_3' of C_1', C_2' and C_3' if the radii R_1, R_2 and R_3 of the given circles C_1, C_2 and C_3 (which fix the figure of these circles) are given. For this purpose we let the objects in Figure 1 undergo an arbitrary inversion. Let O be the center of inversion and we choose a rectangular grid with O as its origin and such that ℓ_2 and ℓ_3 are parallel to the $x-$axis. For the power of inversion we can without any objection choose the unit. The inversion is then given by

$$x' = \frac{x}{x^2+y^2}, \qquad y' = \frac{y}{x^2+y^2}.$$

From this it is clear that the circle with center (x_0, y_0) and radius ρ is transformed into a circle of radius

$$\left|\frac{\rho}{x_0^2+y_0^2-\rho^2}\right|.$$

[4]See Figure 2 in the Appendix, which we add in the present translation.

If the coordinates of the center of K are (a,b), then those of K_1 are $(a+4r\sqrt{2},\ b)$. From this it follows that

$$R_1 = \left|\frac{2r}{a^2+b^2-4r^2}\right|, \qquad R_1' = \left|\frac{2r}{(a+4r\sqrt{2})^2+b^2-4r^2}\right|.$$

The lines ℓ_2 and ℓ_3 are inverted into circles of radii

$$R_2 = \frac{1}{2|b-2r|}, \qquad R_3 = \frac{1}{2|b+2r|}.$$

Now we first assume that O is chosen between ℓ_2 and ℓ_3, and outside K. The circles C_1, C_2 and C_3 then are pairwise tangent *externally*. One has $b-2r<0$, $b+2r>0$, and $a^2+b^2>4r^2$, so that

$$R_2 = \frac{1}{2(2r-b)}, \qquad R_3 = \frac{1}{2(2r+b)}, \qquad R_1 = \frac{2r}{a^2+b^2-4r^2}.$$

Consequently,

$$a = \pm\frac{1}{2}\sqrt{\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}}, \quad b = \frac{1}{4}\left(\frac{1}{R_3}-\frac{1}{R_2}\right), \quad r = \frac{1}{8}\left(\frac{1}{R_3}+\frac{1}{R_2}\right),$$

so that one of the solutions has

$$\frac{1}{R_1'} = \frac{1}{R_1}+\frac{2}{R_2}+\frac{2}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},$$

and in the same way

$$\frac{1}{R_2'} = \frac{2}{R_1}+\frac{1}{R_2}+\frac{2}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},$$

$$\frac{1}{R_3'} = \frac{2}{R_1}+\frac{2}{R_2}+\frac{1}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}, \qquad (1)$$

while the second solution is found by replacing the square roots on the right hand sides by their opposites and then taking absolute values. The first solution consists of three circles which are pairwise tangent externally. For the second there are different possibilities. It may consist of three circles tangent to each other externally, or of three circles, two tangent externally, and with a third circle tangent internally to each of them.[5] One can check the correctness of this remark by choosing O outside each of the circles K_1, K_2 and K_3 respectively, or inside these. According as one chooses O on the circumference of one of the circles, or at the point of tangency of two of these circles, respectively one , or two, straight lines[6] appear in the solution.

Finally, if one takes O outside the strip bordered by ℓ_2 and ℓ_3, or inside K, then the resulting circles have two internal and one external tangencies. If the circle C_1 is tangent *internally* to C_2 and C_3, then one should replace in solution (1) R_1 by $-R_1$, and the same for the second solution. In both solutions the circles are tangent

[5]See Figures 2 and 3 in the Appendix.

[6]See Figures 4, 5, and 6 in the Appendix.

to each other externally.[7] Incidentally, one can take (1) and the corresponding expression, where the sign of the square root is taken oppositely, as the general solution for each case, if one agrees to accept also negative values for a radius and to understand that two externally tangent circles have radii of equal signs and internally tangent circles of opposite signs.

There are two circles that are tangent to the three given circles. [8] This also follows immediately from Figure 1. In this figure the radii of these circles are both $2r$, the coordinates of their centers $(a \pm 4r,\ b)$. After inversion one finds for the radii of these 'inscribed' circles of the figure C_1, C_2, C_3:

$$\frac{1}{\rho_{1,2}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \pm 2\sqrt{\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}}, \tag{2}$$

expressions showing great analogy to (1). One finds these already in Steiner [9] (*Werke* I, pp. 61 – 63, with a clarifying remark by Weierstrass, p.524). [10] While ρ_1 is always positive, $\frac{1}{\rho_2}$ can be greater than, equal to, or smaller than zero. One of the circles is tangent to all the given circles externally, the other is tangent to them all externally, or all internally, (or in the transitional case a straight line). One can read these properties easily from Figure 1 as well. Steiner proves (2) by a straightforward calculation with the help of a formula for the altitude of a triangle.

From (1) and (2) one can derive a large number of relations among the radii R_i of the given circles, the radii R_i' of the Malfatti circles, and the radii ρ_i of the tangent circles. We only mention

$$\frac{1}{R_1} + \frac{1}{R_1'} = \frac{1}{R_2} + \frac{1}{R_2'} = \frac{1}{R_3} + \frac{1}{R_3'}.$$

About the formulas (1) we want to make some more remarks. After finding for the figure $\mathcal{S}$ of given circles C_1, C_2, C_3 one of the two sets $\mathcal{S}'$ of Malfatti circles C_1', C_2', C_3', clearly one may repeat the same construction to $\mathcal{S}'$. One of the two sets of Malfatti circles that belong to $\mathcal{S}'$ clearly is $\mathcal{S}$. Continuing this way in two directions *a chain of triads of circles* arises, with the property that each of two consecutive triples is a Malfatti figure of the other.

By *iteration* of formula (1) one can express the radii of the circles in the n^{th} triple in terms of the radii of the circles one begins with. If one applies (1) to $\frac{1}{R_i'}$, and chooses the negative square root, then one gets back $\frac{1}{R_i}$. For the new set we find

$$\frac{1}{R_1''} = \frac{17}{R_1} + \frac{16}{R_2} + \frac{16}{R_3} + 20\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)}$$

[7] See Figure 7 in the Appendix.

[8] See Figure 8 in the Appendix.

[9] Steiner [15].

[10] This formula has become famous in modern times since the appearance of Soddy [5]. See [6]. According to Boyer and Merzbach [2], however, an equivalent formula was already known to René Descartes, long before Soddy and Steiner.

and cyclic permutations. For the next sets,

$$\begin{aligned}\frac{1}{R_1^{(3)}} &= \frac{161}{R_1}+\frac{162}{R_2}+\frac{162}{R_3}+198\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}\\ \frac{1}{R_1^{(4)}} &= \frac{1601}{R_1}+\frac{1600}{R_2}+\frac{1600}{R_3}+1960\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}\end{aligned}$$

If one takes

$$\begin{aligned}\frac{1}{R_1^{(2p)}} &= \frac{a_{2p}+1}{R_1}+\frac{a_{2p}}{R_2}+\frac{a_{2p}}{R_3}+b_{2p}\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}\\ \frac{1}{R_1^{(2p+1)}} &= \frac{a_{2p+1}+1}{R_1}+\frac{a_{2p+1}+2}{R_2}+\frac{a_{2p+1}+2}{R_3}\\ &\quad +b_{2p+1}\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},\end{aligned}$$

then one finds the recurrences [11]

$$\begin{aligned}a_{2p+1} &= 10a_{2p}-a_{2p-1},\\ a_{2p} &= 10a_{2p-1}-a_{2p-2}+16,\\ b_k &= 10b_{k-1}-b_{k-2},\end{aligned}$$

from which one can compute the radii of the circles in the triples.

The figure of three pairwise tangent circles C_1, C_2, C_3 forms with a set of Malfatti circles C_1', C_2', C_3' a configuration of six circles, of which each is tangent to four others. If one maps the circles of the plane to points in a three dimensional projective space, where the point-circles correspond with the points of a quadric surface Ω, then the configuration matches with an octahedron, of which the edges are tangent to Ω. The construction that was under discussion is thus the same as the following problem: *around a quadric surface Ω (for instance a sphere) construct an octahedron, of which the edges are tangent to Ω, and the vertices of one face are given.* This problem therefore has two solutions. And with the above chain corresponds a chain of triangles, all circumscribing Ω, and having the property that two consecutive triangles are opposite faces of a circumscribing octahedron.

From the formulas derived above for the radii it follows that these are decreasing if one goes in one direction along the chain, and increasing in the other direction, a fact that is apparent from the figure. Continuing in one direction, the triple of circles will eventually converge to a single point. With the question of how this point is positioned with respect to the given circles, we wish to end this modest contribution to the knowledge of the curious problem of Malfatti.

[11] These are the sequences A001078 and A053410 in N.J.A. Sloane's *Encyclopedia of Integer Sequences* [13].

Appendix

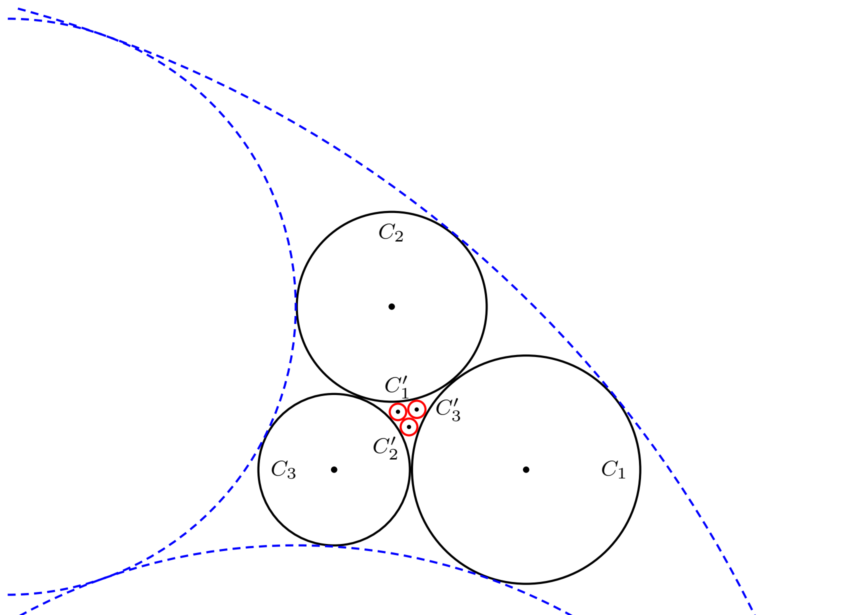

Figure 2

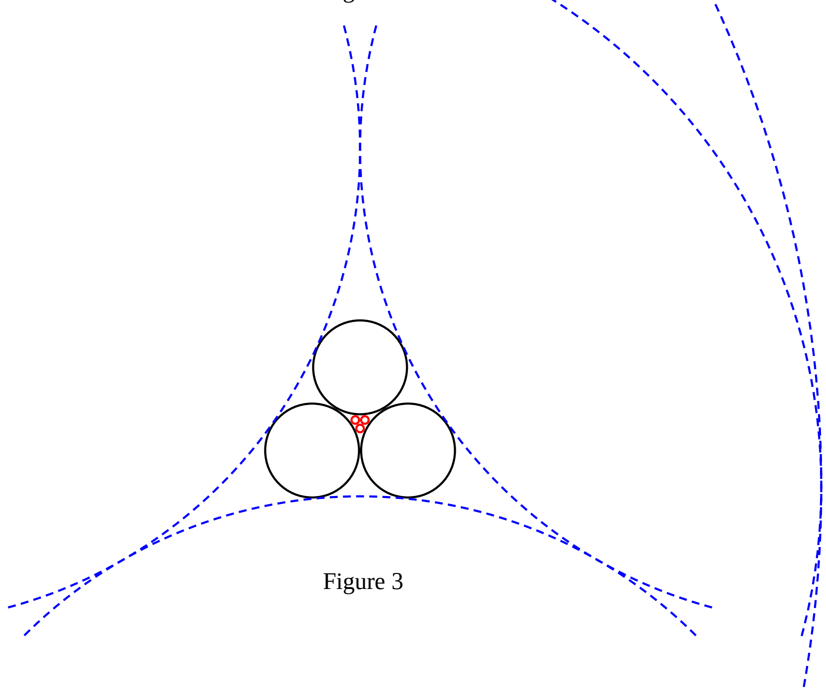

Figure 3

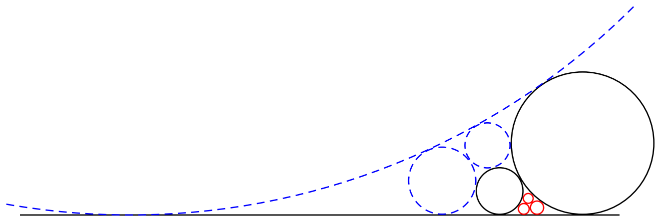

Figure 4

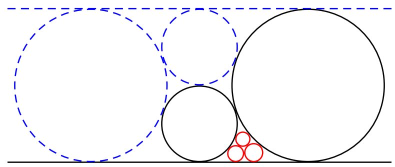

Figure 5

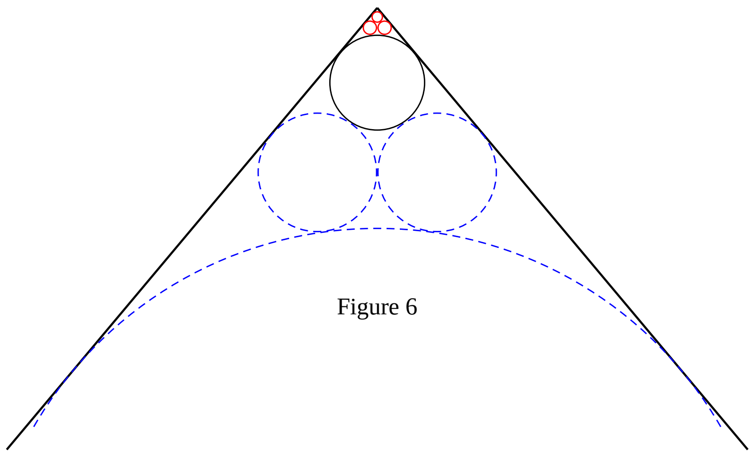

Figure 6

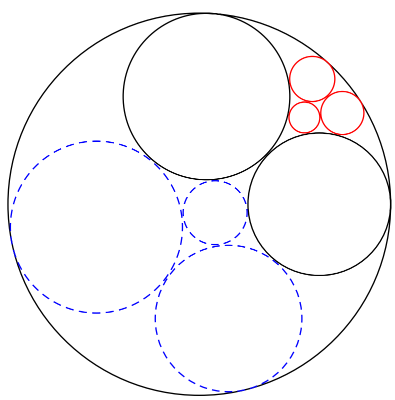

Figure 7

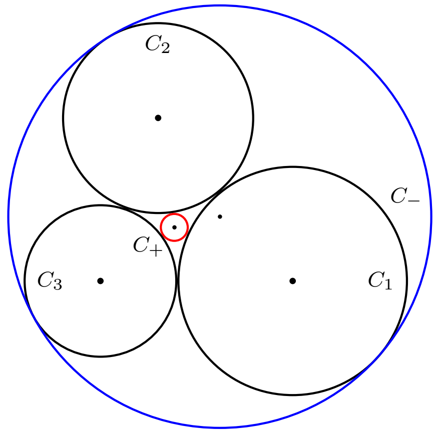

Figure 8

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