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A Morley Configuration

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Abstract. Given a triangle, the isogonal conjugates of the infinite points of the side lines of the Morley (equilateral) triangle is an equilateral triangle PQR inscribed in the circumcircle. Their isotomic conjugates form another equilateral triangle $P'Q'R'$ inscribed in the Steiner circum-ellipse, homothetic to PQR at the Steiner point. We show that under the one-to-one correspondence $P \mapsto P'$ between the circumcircle and the Steiner circum-ellipse established by isogonal and then isotomic conjugations, this is the only case when both PQR and $P'Q'R'$ are equilateral.

1. Introduction

Consider the Morley triangle $M_aM_bM_c$ of a triangle ABC, the equilateral triangle whose vertices are the intersections of pairs of angle trisectors adjacent to a side. Under *isogonal* conjugation, the infinite points of the Morley lines M_bM_c, M_cM_a, M_aM_b correspond to three points G_a, G_b, G_c on the circumcircle. These three points form the vertices of an equilateral triangle. This phenomenon is true for any three lines making 60° angles with one another.[1]

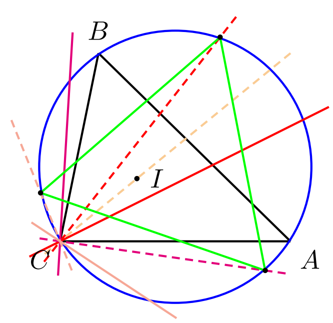

Figure 1

Under *isotomic* conjugation, on the other hand, the infinite points of the same three Morley lines correspond to three points T_a, T_b, T_c on the Steiner circum-ellipse. It is interesting to note that these three points also form the vertices of an equilateral triangle. Consider the mapping that sends a point P to P', the isotomic conjugate of the isogonal conjugate of P. This maps the circumcircle onto the Steiner circum-ellipse. The main result of this paper is that $G_aG_bG_c$ is the only equilateral triangle PQR for which $P'Q'R'$ is also equilateral.

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[1]In Figure 1, the isogonal conjugates of the infinite points of the three lines through A are the intersections of the circumcircle with their reflections in the bisector of angle A.

Main Theorem. Let PQR be an equilateral triangle inscribed in the circumcircle. The triangle $P'Q'R'$ is equilateral if and only if P, Q, R are the isogonal conjugates of the infinite points of the Morley lines.

Before proving this theorem, we make some observations and interesting applications.

2. Homothety of $G_aG_bG_c$ and $T_aT_bT_c$

The two equilateral triangles $G_aG_bG_c$ and $T_aT_bT_c$ are homothetic at the Steiner point S, with ratio of homothety $1 : 4\sin^2\Omega$, where Ω is the Brocard angle of triangle ABC. The circumcircle of the equilateral triangle $T_aT_bT_c$ has center at the third Brocard point [2], the isotomic conjugate of the symmedian point, and is tangent to the circumcircle of ABC at the Steiner point S. In other words, the circle centered at the third Brocard point and passing through the Steiner point intersects the Steiner circum-ellipse at three other points which are the vertices of an equilateral triangle homothetic to the Morley triangle. This circle has radius $4R\sin^2\Omega$ and is smaller than the circumcircle, except when triangle ABC is equilateral.

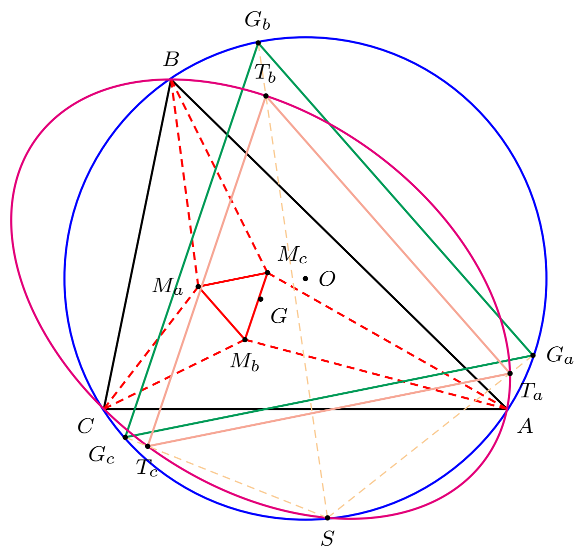

Figure 2

The triangle $G_aG_bG_c$ is the circum-tangential triangle in [3]. It is homothetic to the Morley triangle. From this it follows that the points G_a, G_b, G_c are the points of tangency with the circumcircle of the deltoid which is the envelope of the axes of inscribed parabolas.[3]

[2]This point is denoted by X_{76} in [3].

[3]The axis of an inscribed parabola with focus F is the perpendicular from F to its Simson line, or equivalently, the homothetic image of the Simson line of the antipode of F on the circumcircle, with homothetic center G and ratio -2. In [5], van Lamoen has shown that the points of contact of Simson lines tangent to the nine-point circle also form an equilateral triangle homothetic to the Morley triangle.

3. Equilateral triangles inscribed in an ellipse

Let $\mathcal{E}$ be an ellipse centered at O, and U a point on $\mathcal{E}$. With homothetic center O, ratio $-\frac{1}{2}$, maps U to u. Construct the parallel through u to its polar with respect to $\mathcal{E}$, to intersect the ellipse at V and W. The circumcircle of UVW intersects $\mathcal{E}$ at the Steiner point S of triangle UVW. Let M be the third Brocard point of UVW. The circle, center M, passing through S, intersects $\mathcal{E}$ at three other points which form the vertices of an equilateral triangle. See Figure 3.

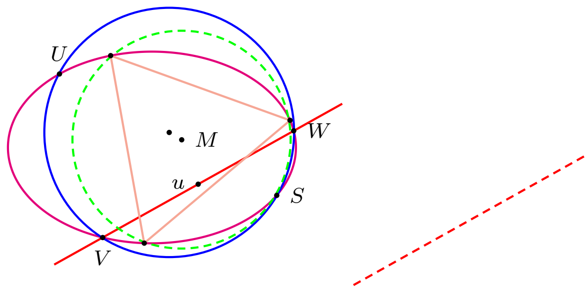

Figure 3

From this it follows that the locus of the centers of equilateral triangles inscribed in the Steiner circum-ellipse of ABC is the ellipse

$$\sum_{\text{cyclic}} a^2(a^2+b^2+c^2)x^2 + (a^2(b^2+c^2) - (2b^4+b^2c^2+2c^4))yz = 0$$

with the same center and axes.

4. Some preliminary results

Proposition 1. *If a circle through the focus of a parabola has its center on the directrix, there exists an equilateral triangle inscribed in the circle, whose side lines are tangent to the parabola.*

Proof. Denote by p the distance from the focus F of the parabola to its directrix. In polar coordinates with the pole at F, let the center of the circle be the point $(\frac{p}{\cos\alpha}, \alpha)$. The radius of the circle is $R = \frac{p}{\cos\alpha}$. See Figure 4. If this center is at a distance d to the line tangent to the parabola at the point $(\frac{p}{1+\cos\theta}, \theta)$, then

$$\frac{d}{R} = \left|\frac{\cos(\theta-\alpha)}{2\cos\frac{\theta}{2}}\right|.$$

Thus, for $\theta = \frac{2}{3}\alpha$, $\frac{2}{3}(\alpha+\pi)$ and $\frac{2}{3}(\alpha-\pi)$, we have $d = \frac{R}{2}$, and the lines tangent to the parabola at these three points form the required equilateral triangle. □

Proposition 2. *If P lies on the circumcircle, then the line PP' passes through the Steiner point S.* [4]

[4] More generally, if $u+v+w=0$, the line joining $(\frac{p}{u} : \frac{q}{v} : \frac{r}{w})$ to $(\frac{l}{u} : \frac{m}{v} : \frac{n}{w})$ passes through the point $(\frac{1}{qn-rm} : \frac{1}{rl-pn} : \frac{1}{pm-ql})$ which is the fourth intersection of the two circumconics $\frac{p}{u}+\frac{q}{v}+\frac{r}{w}=0$ and $\frac{l}{u}+\frac{m}{v}+\frac{n}{w}=0$.

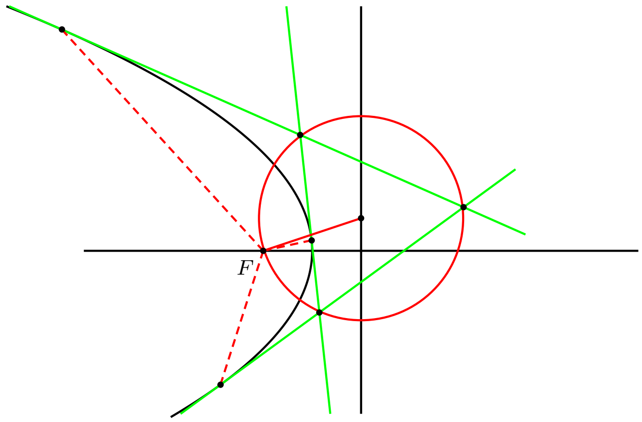

Figure 4

It follows that a triangle PQR inscribed in the circumcircle is always perspective with $P'Q'R'$ (inscribed in the Steiner circum-ellipse) at the Steiner point. The perspectrix is a line parallel to the tangent to the circumcircle at the focus of the Kiepert parabola. [5]

We shall make use of the Kiepert parabola

$$\mathcal{P}: \qquad \sum (b^2-c^2)^2x^2 - 2(c^2-a^2)(a^2-b^2)yz = 0.$$

This is the inscribed parabola with perspector the Steiner point S, focus $\mathcal{S} = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$,[6] and the Euler line as directrix. For more on inscribed parabolas and inscribed conics in general, see [1].

Proposition 3. *Let PQ be a chord of the circumcircle. The following statements are equivalent:* [7]

(a) PQ and $P'Q'$ are parallel.
(b) The line PQ is tangent to the Kiepert parabola $\mathcal{P}$.
(c) The Simson lines $s(P)$ and $s(Q)$ intersect on the Euler line.

Proof. If the line PQ is $ux+vy+wz=0$, then $P'Q'$ is $a^2ux+b^2vy+c^2wz=0$. These two lines are parallel if and only if

$$\frac{b^2-c^2}{u} + \frac{c^2-a^2}{v} + \frac{a^2-b^2}{w} = 0, \qquad (1)$$

which means that PQ is tangent to the the Kiepert parabola.

The common point of the Simson lines $s(P)$ and $s(Q)$ is $(x:y:z)$, where

$$\begin{aligned} x = \; & (2b^2(c^2+a^2-b^2)v + 2c^2(a^2+b^2-c^2)w - (c^2+a^2-b^2)(a^2+b^2-c^2)u) \\ & \cdot((a^2+b^2-c^2)v + (c^2+a^2-b^2)w - 2a^2u), \end{aligned}$$

[5] This line is also parallel to the trilinear polars of the two isodynamic points.

[6] This is the point X_{110} in [3].

[7] These statements are also equivalent to (d): The orthopole of the line PQ lies on the Euler line.

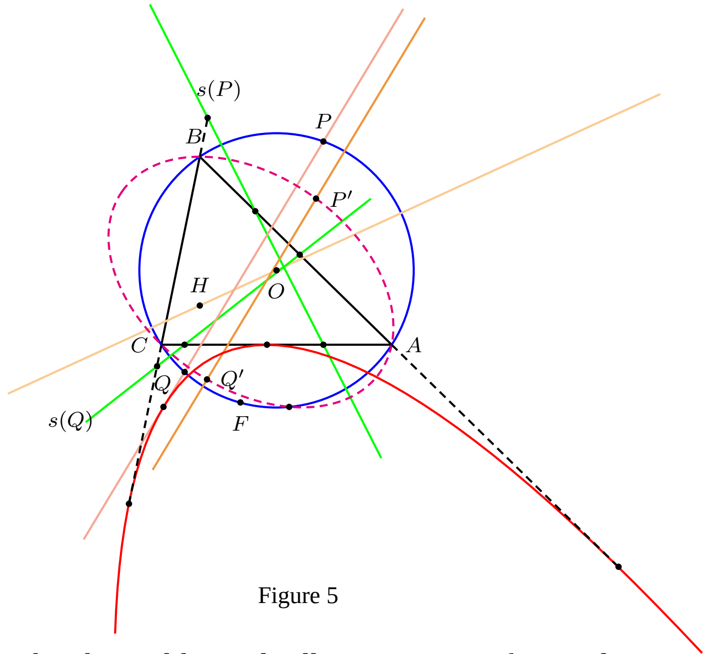

Figure 5

and y and z can be obtained by cyclically permuting a, b, c, and u, v, w. This point lies on the Euler line if and only if (1) is satisfied. □

In the following proposition, (ℓ_1, ℓ_2) denotes the directed angle between two lines ℓ_1 and ℓ_2. This is the angle through which the line ℓ_1 must be rotated in the positive direction in order to become parallel to, or to coincide with, the line ℓ_2. See [2, §§16–19.].

Proposition 4. *Let P, Q, R be points on the circumcircle. The following statements are equivalent.*

(a) The Simson lines $s(P)$, $s(Q)$, $s(R)$ are concurrent.

(b) $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$.

(c) $s(P)$ and QR are perpendicular; so are $s(Q)$ and RP; $s(R)$ and PQ.

Proof. See [4, §§2.16–20]. □

Proposition 5. *A line ℓ is parallel to a side of the Morley triangle if and only if*

$$(AB, \ell) + (BC, \ell) + (CA, \ell) = 0 \pmod{\pi}.$$

Proof. Consider the Morley triangle $M_aM_bM_c$. The line BM_c and CM_b intersecting at P, the triangle PM_bM_c is isoceles and $(M_cM_b, M_cP) = \frac{1}{3}(B + C)$. Thus, $(BC, M_bM_c) = \frac{1}{3}(B - C)$. Similarly, $(CA, M_bM_c) = \frac{1}{3}(C - A) + \frac{\pi}{3}$, and $(AB, M_bM_c) = \frac{1}{3}(A - B) - \frac{\pi}{3}$. Thus

$$(AB, M_bM_c) + (BC, M_bM_c) + (CA, M_bM_c) = 0 \pmod{\pi}.$$

There are only three directions of line ℓ for which $(AB, \ell) + (BC, \ell) + (CA, \ell) = 0$. These can only be the directions of the Morley lines. □

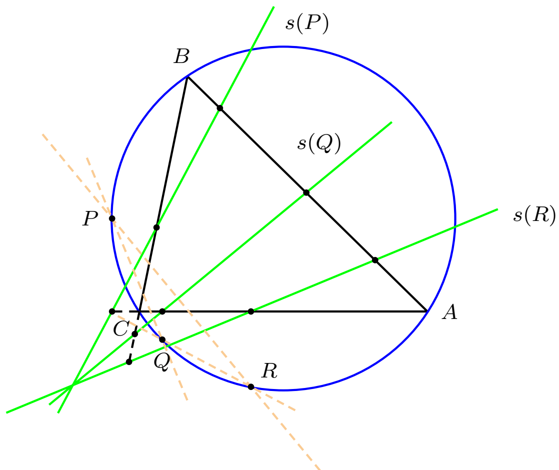

Figure 6

5. Proof of Main Theorem

Let $\mathcal{P}$ be the Kiepert parabola of triangle ABC. By Proposition 1, there is an equilateral triangle PQR inscribed in the circumcircle whose sides are tangent to $\mathcal{P}$. By Propositions 2 and 3, the triangle $P'Q'R'$ is equilateral and homothetic to PQR at the Steiner point S. By Proposition 3 again, the Simson lines $s(P)$, $s(Q)$, $s(R)$ concur. It follows from Proposition 4 that $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$. Since the lines PQ, QR, and RP make 60° angles with each other, we have

$$(AB, PQ) + (BC, PQ) + (CA, PQ) = 0 \pmod{\pi},$$

and PQ is parallel to a side of the Morley triangle by Proposition 5. Clearly, this is the same for QR and RP. By Proposition 4, the vertices P, Q, R are the isogonal conjugates of the infinite points of the Morley sides.

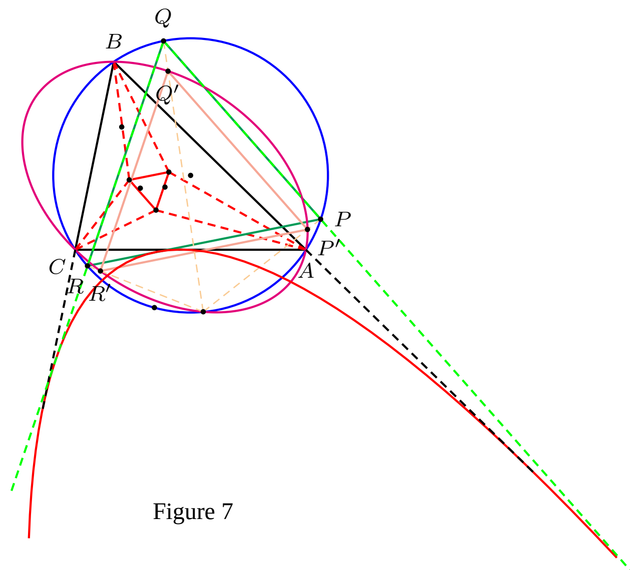

Figure 7

Uniqueness: For $M(x:y:z)$, let

$$f(M) = \frac{x+y+z}{\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}}.$$

The determinant of the affine mapping $P \mapsto P'$, $Q \mapsto Q'$, $R \mapsto R'$ is

$$\frac{f(P)f(Q)f(R)}{a^2b^2c^2}.$$

This determinant is positive for P, Q, R on the circumcircle, which does not intersect the Lemoine axis $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$. Thus, if both triangles are equilateral, the similitude $P \mapsto P'$, $Q \mapsto Q'$, $R \mapsto R'$ is a *direct* one. Hence,

$$(SP', SQ') = (SP, SQ) = (RP, RQ) = (R'P', R'Q'),$$

and the circle $P'Q'R'$ passes through S. Now, through any point on an ellipse, there is a unique circle intersecting the ellipse again at the vertices of an equilateral triangle. This establishes the uniqueness, and completes the proof of the theorem.

6. Concluding remarks

We conclude with a remark and a generalization.

(1) The reflection of $G_aG_bG_c$ in the circumcenter is another equilateral triangle PQR (inscribed in the circumcircle) whose sides are parallel to the Morley lines.[8] This, however, does not lead to an equilateral triangle inscribed in the Steiner circum-ellipse.

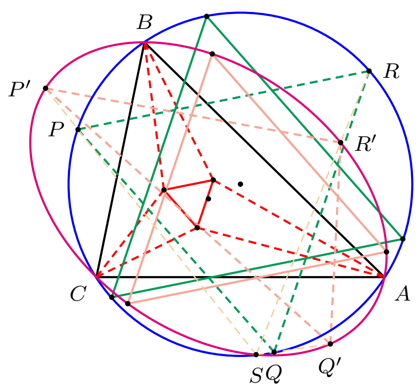

Figure 8

(2) Consider the circum-hyperbola $\mathcal{C}$ through the centroid G and the symmedian point K.[9] For any point P on $\mathcal{C}$, let $\mathcal{C}_P$ be the circumconic with perspector P, intersecting the circumcircle again at a point S_P.[10] For every point M on the

[8]This is called the circumnormal triangle in [3].

[9]The center of this hyperbola is the point $(a^4(b^2-c^2)^2 : b^4(c^2-a^2)^2 : c^4(a^2-b^2)^2)$.

[10]The perspector of a circumconic is the perspector of the triangle bounded by the tangents to the conic at the vertices of ABC. If $P=(u:v:w)$, the circumconic $\mathcal{C}_P$ has center $(u(v+w-u) : v(w+u-v) : w(u+v-w))$, and S_P is the point $\left(\frac{1}{b^2w-c^2v} : \frac{1}{c^2u-a^2w} : \frac{1}{a^2v-b^2u}\right)$. See Footnote 4.

circumcircle, denote by M' the second common point of $\mathcal{C}_U$ and the line MS_P. Then, if G_a, G_b, G_c are the isogonal conjugates of the infinite points of the Morley lines, $G'_aG'_bG'_c$ is homothetic to $G_aG_bG_c$ at S_U. The reason is simple: Proposition 3 remains true. For $U = G$, this gives the equilateral triangle $T_aT_bT_c$ inscribed in the case of the Steiner circum-ellipse. Here is an example. For $U = (a(b+c) : b(c+a) : c(a+b))$, [11] we have the circumellipse with center the Spieker center $(b+c : c+a : a+b)$. The triangles $G_aG_bG_c$ and $G'_aG'_bG'_c$ are homothetic at $X_{100} = (\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b})$, and the circumcircle of $G'_aG'_bG'_c$ is the incircle of the anticomplementary triangle, center the Nagel point, and ratio of homothety $R : 2r$.

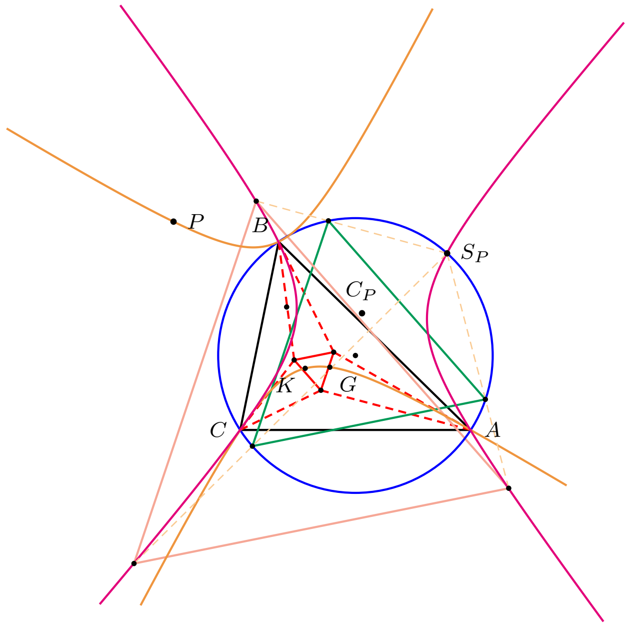

Figure 9

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[11] This is the point X_{37} in [3].