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Some Remarkable Concurrences

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Abstract. In May 1999, Steve Sigur, a high school teacher in Atlanta, Georgia, posted on The Math Forum, a notice stating that one of his students (Josh Klehr) noticed that

> *Given a triangle with mid-point of each side. Through each mid-point, draw a line whose slope is the reciprocal to the slope of the side through that mid-point. These lines concur.*

Sigur then stated that "we have proved this". Here, we extend this result to the case where the slope of the line through the mid-point is a constant times the reciprocal of the slope of the side.

In May 1999, Steve Sigur, a high school teacher in Atlanta, Georgia, posted on The Math Forum, a notice stating that one of his students (Josh Klehr) noticed that

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There was a further statement that another student (Adam Bliss) followed up with a result on the concurrency of reflected line, with the point of concurrency lying on the nine-point circle. This was subsequently proved by Louis Talman [2]. See also the variations, using the feet of the altitudes in place of the mid-points and different reflections in the recent paper by Floor van Lamoen [1].

Here, we are interested in a generalization of Klehr's result.

At the mid-point of each side of a triangle, we construct the line such that the product of the slope of this line and the slope of the side of the triangle is a fixed constant. To make this clear, the newly created lines have slopes of the fixed constant times the reciprocal of the slopes of the sides of the triangle with respect to a given line (parallel to the x–axis used in the Cartesian system). We show that the three lines obtained are always concurrent.

Further, the locus of the points of concurrency is a rectangular hyperbola. This hyperbola intersects the side of the triangles at the mid-points of the sides, and each side at another point. These three other points, when considered with the vertices of the triangle opposite to the point, form a Ceva configuration. Remarkably, the point of concurrency of these Cevians lies on the circumcircle of the original triangle.

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Since we are dealing with products of slopes, we have restricted ourselves to a Cartesian proof.

Suppose that we have a triangle with vertices $(0,0)$, $(2a,2b)$ and $(2c,2d)$.

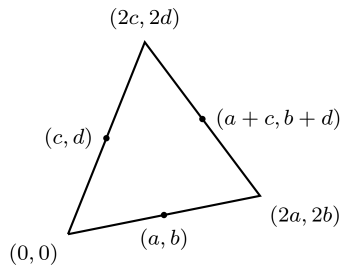

Figure 1

In order to ensure that the triangle is not degenerate, we assume that $ad-bc \neq 0$. For ease of proof, we also take $0 \neq a \neq c \neq 0$ and $0 \neq b \neq d \neq 0$ to avoid division by zero. However, by continuity, the results obtained here can readily be extended to include these cases.

At the mid-point of each side, we find the equations of the new lines:

Mid-point	Slope	Equation
(a,b)	$\frac{\lambda a}{b}$	$y = \frac{\lambda a}{b}x + \frac{b^2 - \lambda a^2}{b}$
(c,d)	$\frac{\lambda c}{d}$	$y = \frac{\lambda c}{d}x + \frac{d^2 - \lambda c^2}{d}$
$(a+c, b+d)$	$\frac{\lambda(c-a)}{d-b}$	$y = \frac{\lambda(c-a)}{d-b}x + \frac{(a^2-c^2)\lambda + (d^2-b^2)}{d-b}$

With the aid of a computer algebra program, we find that the first two lines meet at

$$\left(\frac{\lambda(a^2d - bc^2) + bd(d-b)}{\lambda(ad-bc)}, \frac{\lambda ac(a-c) + (ad^2 - b^2c)}{(ad-bc)}\right),$$

which it is easy to verify lies on the third line.

By eliminating λ from the equations

$$x = \frac{\lambda(a^2d - bc^2) + bd(d-b)}{\lambda(ad-bc)}, \qquad y = \frac{\lambda ac(a-c) + (ad^2 - b^2c)}{(ad-bc)},$$

we find that the locus of the points of concurrency is

$$y = \frac{abcd(a-c)(d-b)}{ad-bc} \cdot \frac{1}{(ad-bc)x - (a^2d - bc^2)} + \frac{ad^2 - b^2c}{ad-bc}.$$

This is a rectangular hyperbola, with asymptotes

$$x = \frac{a^2d - bc^2}{ad-bc}, \qquad y = \frac{ad^2 - b^2c}{ad-bc}.$$

Now, this hyperbola meets the sides of the given triangle as follows:

from	to	mid-point	new-point
$(0,0)$	$(2a,2b)$	(a,b)	$\left(\frac{ad+bc}{b}, \frac{ad+bc}{a}\right)$
$(0,0)$	$(2c,2d)$	(c,d)	$\left(\frac{ad+bc}{d}, \frac{ad+bc}{c}\right)$
$(2a,2b)$	$(2c,2d)$	$(a+c,b+d)$	$\left(\frac{ad-bc}{d-b}, \frac{ad-bc}{a-c}\right)$

The three lines joining the three points (new-point, in each case) to the vertices opposite are concurrent! (Again, easily shown by computer algebra.) The point of concurrency is

$$\left(2(a-c)\left(\frac{ad+bc}{ad-bc}\right),\ 2(d-b)\left(\frac{ad+bc}{ad-bc}\right)\right).$$

It is easy to check that this point is not on the hyperbola. However, it is also easy to check that this point lies on the circumcircle of the original triangle. (Compare this result with the now known result that the point on the hyperbola corresponding to $\lambda = 1$ lies on the nine-point circle. See [2].)

In Figures 2, 3, 4 below, we illustrate the original triangle ABC, the rectangular hyperbola $YWLPXQVZ$ (where $\lambda < 0$) and $MSOUN$ (where $\lambda > 0$), the asymptotes (dotted lines), the circumcircle and the nine-point circle, and the first remarkable point K.

Figure 2 shows various lines through the mid-points of the sides being concurrent on the hyperbola, and also the concurrency of AX, BY, CZ at K.

Figure 3 shows the lines concurrent through the second remarkable point J, where we join points with parameters λ and $-\lambda$. This point J is indeed the center of the rectangular hyperbola.

Figure 4 shows the parallel lines (or lines concurrent at infinity), where we join points with parameters λ and $-\frac{1}{\lambda}$.

Now, this is a purely Cartesian demonstration. As a result, there are several questions that I have not (yet) answered:

(1) Does the Cevian intersection point have any particular significance?
(2) What significant differences (if any) would occur if the triangle were to be rotated about the origin?
(3) Are there variations of these results along the lines of Floor van Lamoen's paper [1]?

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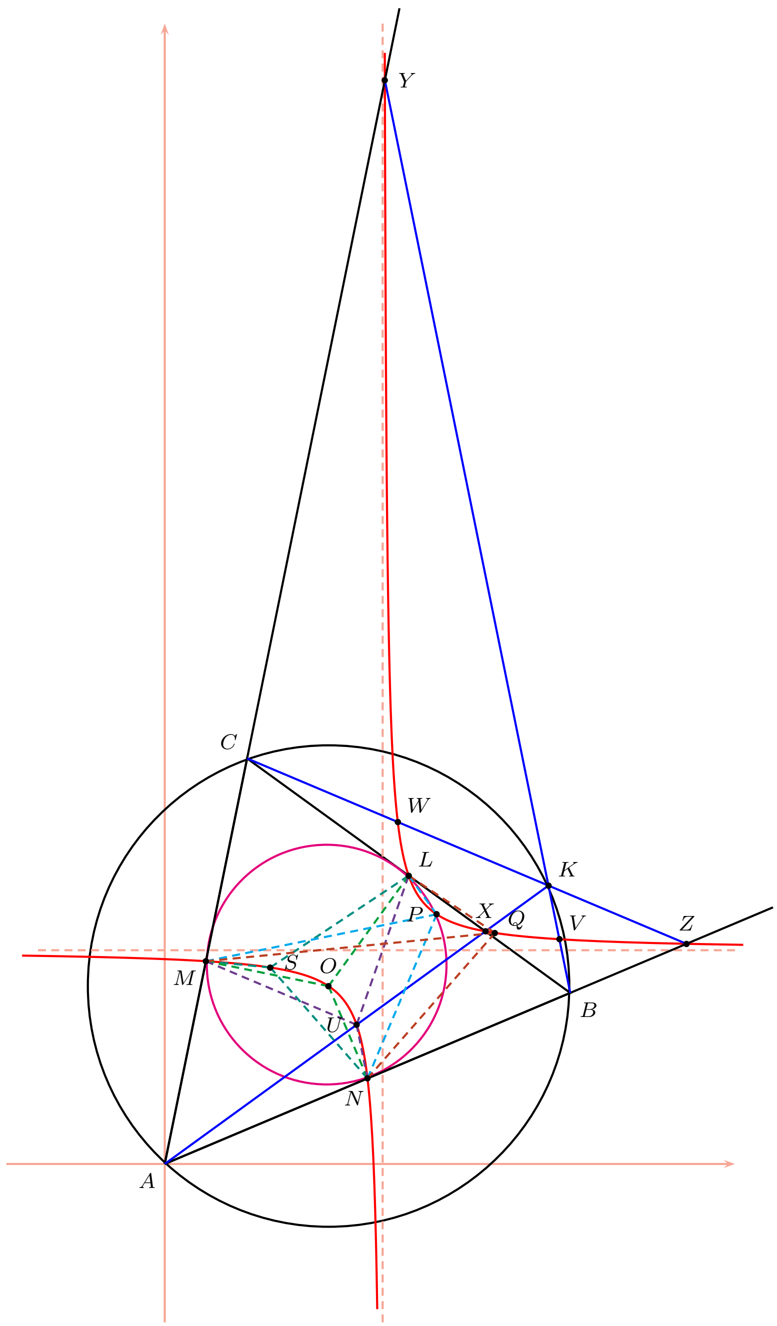

Figure 2

λ	1	$\frac{ac}{bd}$	$\frac{bd}{ac}$	$\frac{b(b-d)}{a(a-c)}$	$\frac{d(b-d)}{c(a-c)}$	$\frac{(b-d)^2}{(a-c)^2}$	$\frac{d^2}{c^2}$	$\frac{b^2}{a^2}$
Point from λ	P	Q	L	M	N	X	Y	Z
Point from $-\lambda$	O	S	U	V	W			

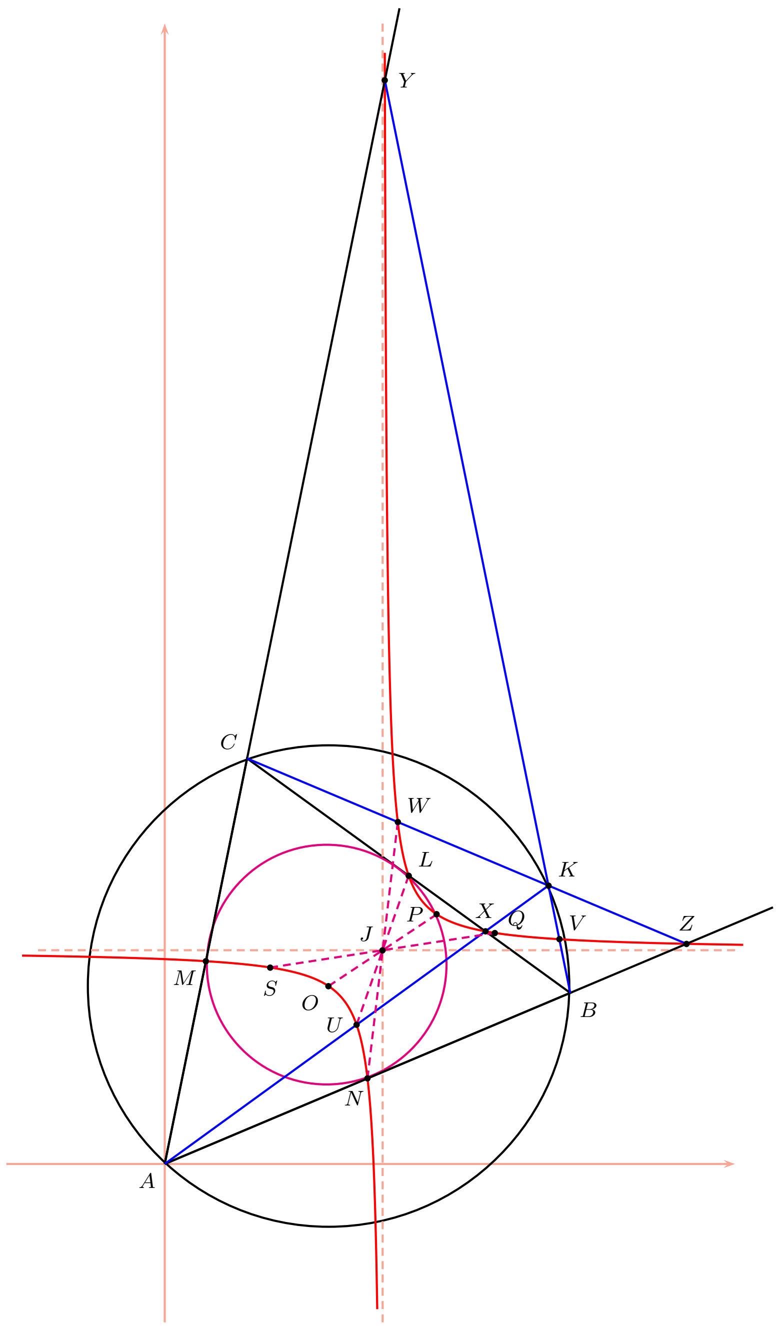

Figure 3

λ	1	$\frac{ac}{bd}$	$\frac{bd}{ac}$	$\frac{b(b-d)}{a(a-c)}$	$\frac{d(b-d)}{c(a-c)}$	$\frac{(b-d)^2}{(a-c)^2}$	$\frac{d^2}{c^2}$	$\frac{b^2}{a^2}$
Point from λ	P	Q	L	M	N	X	Y	Z
Point from $-\lambda$	O	S	U	V	W			

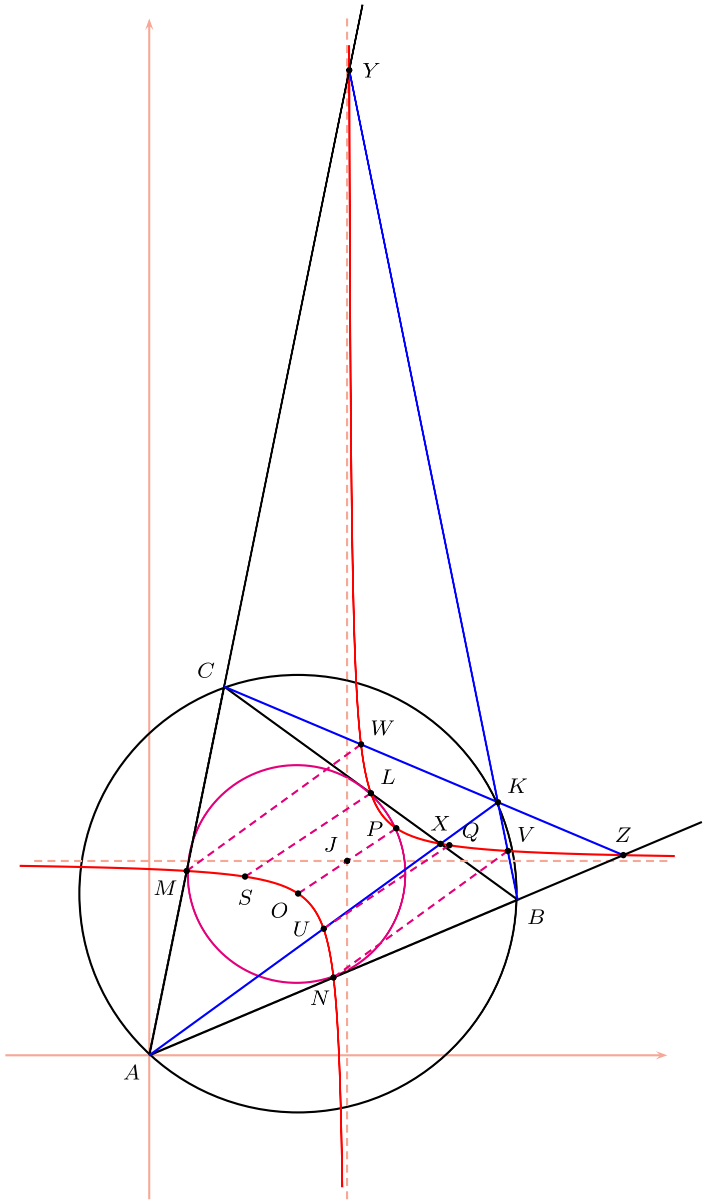

Figure 4

References

[1] F. M. van Lamoen, Morley related triangles on the nine-point circle, *Amer. Math. Monthly*, 107 (2000) 941–945.

[2] L. A. Talman, A remarkable concurrence, http://clem.mscd.edu/∼talmanl, 1999.

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