

Some Properties of the Lemoine Point

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Abstract. The Lemoine point, K , of $\triangle ABC$ has special properties involving centroids of pedal triangles. These properties motivate a definition of Lemoine field, F , and a coordinatization of the plane of $\triangle ABC$ using perpendicular axes that pass through K . These principal axes are symmetrically related to two other lines: one passing through the isodynamic centers, and the other, the isogonic centers.

1. Introduction

Let $A'B'C'$ be the pedal triangle of an arbitrary point Z in the plane of a triangle ABC , and consider the vector field \mathbf{F} defined by $\mathbf{F}(Z) = \mathbf{ZA}' + \mathbf{ZB}' + \mathbf{ZC}'$. It is well known that $\mathbf{F}(Z)$ is the zero vector if and only if Z is the Lemoine point, K , also called the symmedian point. We call \mathbf{F} the *Lemoine field* of $\triangle ABC$ and K the *balance point* of \mathbf{F} .

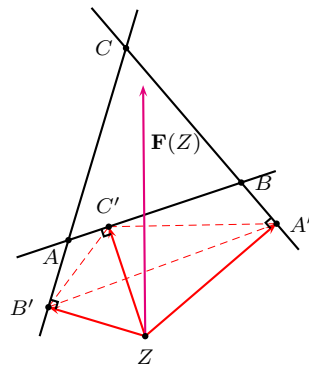


Figure 1

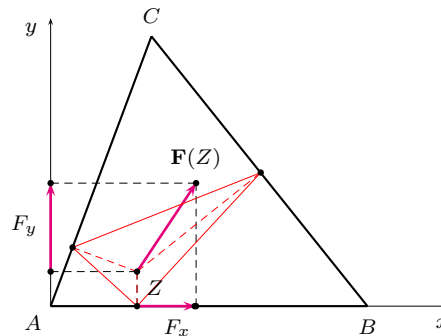


Figure 2

The Lemoine field may be regarded as a physical force field. Any point Z in this field then has a natural motion along a certain curve, or trajectory. See Figure 1. We shall determine parametric equations for these trajectories and find, as a result, special properties of the lines that bisect the angles between the line of the isogonic centers and the line of the isodynamic centers of $\triangle ABC$.

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Я благодарю дорогую Лену за оказанную мне моральную поддержку. Также я очень признателен профессору Кимберлингу за разрешение многочисленных проблем, касающихся английского языка. The author dedicates his work to *Helen* and records his appreciation to the Communicating Editor for assistance in translation.

2. The Lemoine equation

In the standard cartesian coordinate system, place $\triangle ABC$ so that $A = (0, 0)$, $B = (c, 0)$, $C = (m, n)$, and write $Z = (x, y)$. For any line $Px + Qy + R = 0$, the vector H from Z to the projection of Z on the line has components

$$h_x = \frac{-P}{P^2 + Q^2}(Px + Qy + R), \quad h_y = \frac{-Q}{P^2 + Q^2}(Px + Qy + R).$$

From these, one find the components of the three vectors whose sum defines $\mathbf{F}(Z)$:

vector	x - component	y - component
\mathbf{ZA}'	$\frac{-n(nx+y(c-m)-cn)}{n^2+(c-m)^2}$	$\frac{(m-c)(nx+y(c-m)-cn)}{n^2+(c-m)^2}$
\mathbf{ZB}'	$\frac{-n(nx-my)}{m^2+n^2}$	$\frac{m(nx-my)}{m^2+n^2}$
\mathbf{ZC}'	0	$-y$

The components of the Lemoine field $\mathbf{F}(Z) = \mathbf{ZA}' + \mathbf{ZB}' + \mathbf{ZC}'$ are given by

$$F_x = -(\alpha x + \beta y) + d_x, \quad F_y = -(\beta x + \gamma y) + d_y,$$

where

$$\begin{aligned} \alpha &= \frac{n^2}{m^2+n^2} + \frac{n^2}{n^2+(c-m)^2}, & \beta &= \frac{-mn}{m^2+n^2} + \frac{n(c-m)}{n^2+(c-m)^2}, \\ \gamma &= 1 + \frac{m^2}{m^2+n^2} + \frac{(c-m)^2}{n^2+(c-m)^2}; \\ d_x &= \frac{cn^2}{n^2+(c-m)^2}, & d_y &= \frac{cn(c-m)}{n^2+(c-m)^2}. \end{aligned}$$

See Figure 2. Assuming a unit mass at each point Z , Newton's Second Law now gives a system of differential equations:

$$x'' = -(\alpha x + \beta y) + d_x, \quad y'' = -(\beta x + \gamma y) + d_y,$$

where the derivatives are with respect to time, t . We now translate the origin from $(0, 0)$ to the balance point (d_x, d_y) , which is the Lemoine point K , thereby obtaining the system

$$x'' = -(\alpha x + \beta y), \quad y'' = -(\beta x + \gamma y),$$

which has the matrix form

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -M \begin{pmatrix} x \\ y \end{pmatrix}, \quad (1)$$

where $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$. We shall refer to (1) as the *Lemoine equation*.

3. Eigenvalues of the matrix M

In order to solve equation (1), we first find eigenvalues λ_1 and λ_2 of M . These are the solutions of the equation $|M - \lambda I| = 0$, i.e., $(\alpha - \lambda)(\gamma - \lambda) - \beta^2 = 0$, or

$$\lambda^2 - (\alpha + \gamma)\lambda + (\alpha\gamma - \beta^2) = 0.$$

Thus

$$\lambda_1 + \lambda_2 = \alpha + \gamma = 1 + \frac{m^2 + n^2}{m^2 + n^2} + \frac{n^2 + (c - m)^2}{n^2 + (c - m)^2} = 3.$$

Writing a, b, c for the sidelengths $|BC|, |CA|, |AB|$ respectively, we find the determinant

$$|M| = \alpha\gamma - \beta^2 = \frac{n^2}{a^2b^2}(a^2 + b^2 + c^2) > 0.$$

The discriminant of the characteristic equation $\lambda^2 - (\alpha + \gamma)\lambda + (\alpha\gamma - \beta^2) = 0$ is given by

$$D = (\alpha + \gamma)^2 - 4(\alpha\gamma - \beta^2) = (\alpha - \gamma)^2 + 4\beta^2 \geq 0. \quad (2)$$

Case 1: equal eigenvalues $\lambda_1 = \lambda_2 = \frac{3}{2}$. In this case, $D = 0$ and (2) yields $\beta = 0$ and $\alpha = \gamma$. To reduce notation, write $p = c - m$. Then since $\beta = 0$, we have $\frac{m}{m^2+n^2} = \frac{p}{p^2+n^2}$, so that

$$(m - p)(mp - n^2) = 0. \quad (3)$$

Also, since $\alpha = \gamma$, we find after mild simplifications

$$n^4 - (m^2 + p^2)n^2 - 3m^2p^2 = 0. \quad (4)$$

Equation (3) implies that $m = p$ or $mp = n^2$. If $m = p$, then equation (4) has solutions $n = \sqrt{3}m = \sqrt{3}p$. Consequently, $C = \left(\frac{1}{2}c, \frac{\sqrt{3}}{2}c\right)$, so that $\triangle ABC$ is equilateral. However, if $mp = n^2$, then equation (4) leads to $(m + p)^2 = 0$, so that $c = 0$, a contradiction. Therefore from equation (3) we obtain this conclusion: *if the eigenvalues are equal, then $\triangle ABC$ is equilateral.*

Case 2: distinct eigenvalues $\lambda_{1,2} = \frac{3 \pm \sqrt{D}}{2}$. Here $D > 0$, and $\lambda_{1,2} > 0$ according to (2). We choose to consider the implications when

$$\beta = 0, \quad \alpha \neq \gamma. \quad (5)$$

We omit an easy proof that these conditions correspond to $\triangle ABC$ being a right triangle or an isosceles triangle. In the former case, write $c^2 = a^2 + b^2$. Then the characteristic equation yields eigenvalues α and γ , and

$$\alpha = \frac{n^2}{b^2} + \frac{n^2}{a^2} = \frac{n^2(a^2 + b^2)}{a^2b^2} = \frac{n^2c^2}{a^2b^2} = 1,$$

since $ab = nc =$ twice the area of the right triangle. Since $\alpha + \gamma = 3$, $\gamma = 2$.

4. General solution of the Lemoine equation

According to a well known theorem of linear algebra, rotation of the coordinate system about K gives the system $x'' = -\lambda_1 x$, $y'' = -\lambda_2 y$. Let us call the axes of this coordinate system the *principal axes* of the Lemoine field.

Note that if $\triangle ABC$ is a right triangle or an isosceles triangle (cf. conditions (5)), then the angle of rotation is zero, and K is on an altitude of the triangle. In this case, one of the principal axes is that altitude, and the other is parallel to the

corresponding side. Also if $\triangle ABC$ is a right triangle, then K is the midpoint of that altitude.

In the general case, the solution of the Lemoine equation is given by

$$x = c_1 \cos \omega_1 t + c_2 \sin \omega_2 t, \quad y = c_3 \cos \omega_1 t + c_4 \sin \omega_2 t, \quad (6)$$

where $\omega_1 = \sqrt{\lambda_1}$, $\omega_2 = \sqrt{\lambda_2}$. Initial conditions $x(0) = x_0$, $y(0) = y_0$, $x'(0) = 0$, $y'(0) = 0$ reduce (6) to

$$x = x_0 \cos \omega_1 t, \quad y = y_0 \cos \omega_2 t, \quad (7)$$

with $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1^2 + \omega_2^2 = 3$. Equations (7) show that each trajectory is bounded. If $\lambda_1 = \lambda_2$, then the trajectory is a line segment; otherwise, (7) represents a Lissajous curve or an almost-everywhere rectangle-filling curve, according as $\frac{\omega_1}{\omega_2}$ is rational or not.

5. Lemoine sequences and centroidal orbits

Returning to the Lemoine field, \mathbf{F} , suppose Z_0 is an arbitrary point, and G_{Z_0} is the centroid of the pedal triangle of Z_0 . Let Z'_0 be the point to which \mathbf{F} translates Z_0 . It is well known that G_{Z_0} lies on the line $Z_0 Z'_0$ at a distance $\frac{1}{3}$ of that from Z_0 to Z'_0 . With this in mind, define inductively the *Lemoine sequence* of Z_0 as the sequence (Z_0, Z_1, Z_2, \dots) , where Z_n , for $n \geq 1$, is the centroid of the pedal triangle of Z_{n-1} . Writing the centroid of the pedal triangle of Z_0 as $Z_1 = (x_1, y_1)$, we obtain $3(x_1 - x_0) = -\lambda_1 x_0$ and

$$x_1 = \frac{1}{3}(3 - \lambda_1)x_0 = \frac{1}{3}\lambda_2 x_0; \quad y_1 = \frac{1}{3}\lambda_1 y_0.$$

Accordingly, the Lemoine sequence is given with respect to the principal axes by

$$Z_n = \left(x_0 \left(\frac{\lambda_2}{3} \right)^n, y_0 \left(\frac{\lambda_1}{3} \right)^n \right). \quad (8)$$

Since $\frac{1}{3}\lambda_1$ and $\frac{1}{3}\lambda_2$ are between 0 and 1, the points Z_n approach $(0, 0)$ as $n \rightarrow \infty$. That is, the Lemoine sequence of every point converges to the Lemoine point.

Representation (8) shows that Z_n lies on the curve $(x, y) = (x_0 u^t, y_0 v^t)$, where $u = \frac{1}{3}\lambda_2$ and $v = \frac{1}{3}\lambda_1$. We call this curve the *centroidal orbit* of Z_0 . See Figure 3. Reversing the directions of axes if necessary, we may assume that $x_0 > 0$ and $y_0 > 0$, so that elimination of t gives

$$\frac{y}{y_0} = \left(\frac{x}{x_0} \right)^k, \quad k = \frac{\ln v}{\ln u}. \quad (9)$$

Equation (9) expresses the centroidal orbit of $Z_0 = (x_0, y_0)$. Note that if $\omega_1 = \omega_2$, then $v = u$, and the orbit is a line. Now let X_Z and Y_Z be the points in which line ZG_Z meets the principal axes. By (8),

$$\frac{|ZG_Z|}{|G_Z X_Z|} = \frac{\lambda_2}{\lambda_1}, \quad \frac{|ZG_Z|}{|G_Z Y_Z|} = \frac{\lambda_1}{\lambda_2}. \quad (10)$$

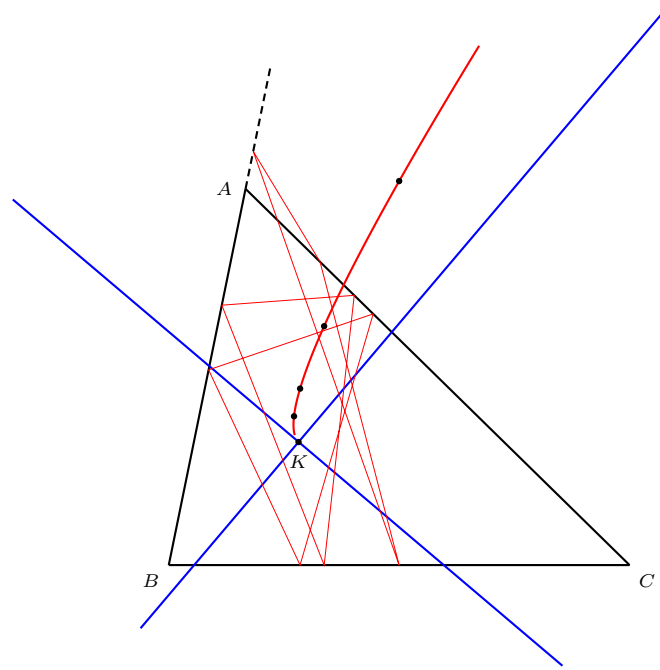


Figure 3

These equations imply that if $\triangle ABC$ is equilateral with center O , then the centroid G_Z is the midpoint of segment OG_Z .

As another consequence of (10), suppose $\triangle ABC$ is a right triangle; let H be the line parallel to the hypotenuse and passing through the midpoint of the altitude H' to the hypotenuse. Let X and Y be the points in which line ZG_Z meets H and H' , respectively. Then $|ZG_Z| : |XG_Z| = |YG_Z| : |ZG_Z| = 2 : 1$.

6. The principal axes of the Lemoine field

Physically, the principal axes may be described as the locus of points in the plane of $\triangle ABC$ along which the “direction” of the Lemoine sequence remains constant. That is, if Z_0 lies on one of the principal axes, then all the points Z_1, Z_2, \dots lie on that axis also.

In this section, we turn to the geometry of the principal axes. Relative to the coordinate system adopted in §5, the principal axes have equations $x = 0$ and $y = 0$. Equation (8) therefore shows that if Z_0 lies on one of these two perpendicular lines, then Z_n lies on that line also, for all $n \geq 1$.

Let A_1, A_2 denote the isodynamic points, and F_1, F_2 the isogonic centers, of $\triangle ABC$. Call lines A_1A_2 and F_1F_2 the *isodynamic axis* and the *isogonic axis* respectively.¹

Lemma 1. *Suppose Z and Z' are a pair of isogonal conjugate points. Let O and O' be the circumcircles of the pedal triangles of Z and Z' . Then $O = O'$, and the center of O is the midpoint between Z and Z' .*

¹The points F_1, F_2, A_1, A_2 are indexed as $X_{13}, X_{14}, X_{15}, X_{16}$ and discussed in [2].

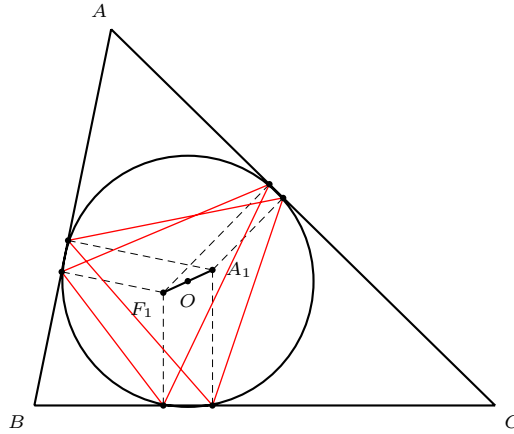


Figure 4

A proof is given in Johnson [1, pp.155–156]. See Figure 4.

Now suppose that $Z = A_1$. Then $Z' = F_1$, and, according to Lemma 1, the pedal triangles of Z and Z' have the same circumcircle, whose center O is the midpoint between A_1 and F_1 . Since the pedal triangle of A_1 is equilateral, the point O is the centroid of the pedal triangle of A_1 .

Next, suppose L is a line not identical to either of the principal axes. Let L' be the reflection of L about one of the principal axes. Then L' is also the reflection of L about the other principal axis. We call L and L' a *symmetric pair of lines*.

Lemma 2. *Suppose that G_P is the centroid of the pedal triangle of a point P , and that Q is the reflection of P in G_P . Then there exists a symmetric pair of lines, one passing through P and the other passing through Q .*

Proof. With respect to the principal axes, write $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. Then $G_P = (\frac{1}{3}\lambda_2 x_P, \frac{1}{3}\lambda_1 y_P)$, and $\frac{2}{3}\lambda_2 x_P = x_P + x_Q$, so that

$$x_Q = \left(\frac{2}{3}\lambda_2 - 1\right)x_P = \frac{1}{3}(2\lambda_2 - (\lambda_1 + \lambda_2))x_P = \frac{1}{3}(\lambda_2 - \lambda_1)x_P.$$

Likewise, $y_Q = \frac{1}{3}y_P(\lambda_1 - \lambda_2)$. It follows that $\frac{x_P}{y_P} = -\frac{x_Q}{y_Q}$. This equation shows that the line $y = \frac{y_P}{x_P} \cdot x$ passing through P and the line $y = \frac{y_Q}{x_Q} \cdot x$ passing through Q are symmetric about the principal axes $y = 0$ and $x = 0$. See Figure 5. \square

Theorem. *The principal axes of the Lemoine field are the bisectors of the angles formed at the intersection of the isodynamic and isogonic axes in the Lemoine point.*

Proof. In Lemma 2, take $P = A_1$ and $Q = F_1$. The symmetric pair of lines are then the isodynamic and isogonic axes. Their symmetry about the principal axes is equivalent to the statement that these axes are the asserted bisectors. \square

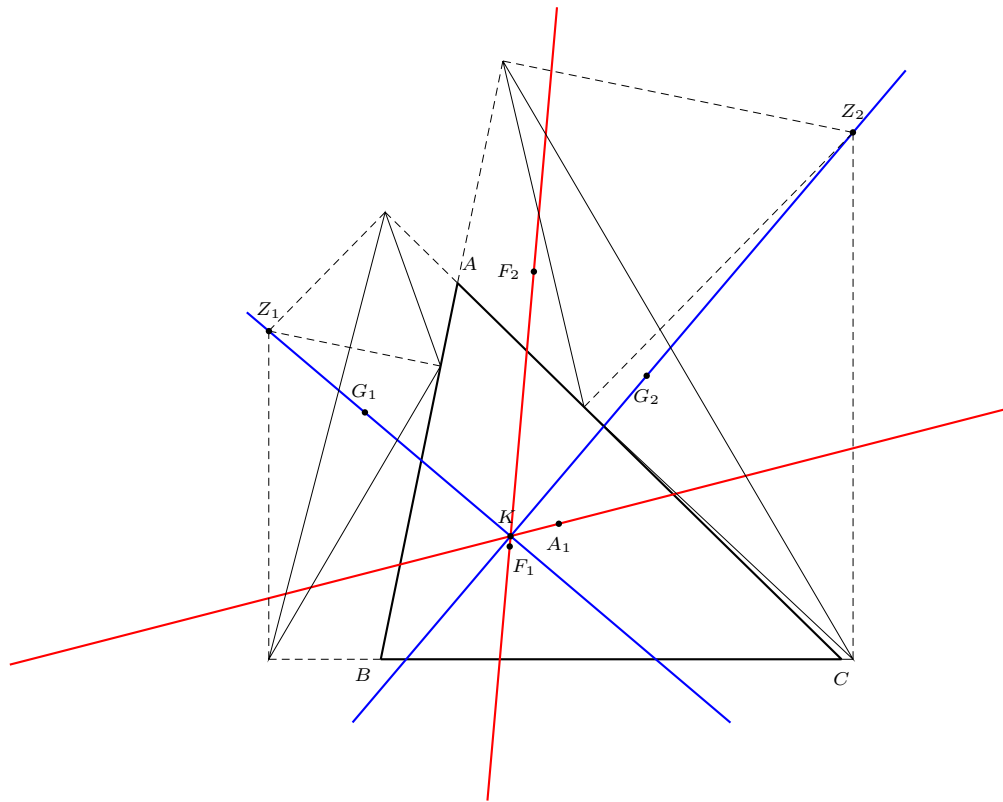


Figure 5

References

- [1] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.
- [2] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000,
<http://cedar.evansville.edu/~ck6/encyclopedia/>.

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