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Multiplying and Dividing Curves by Points

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Abstract. Pointwise products and quotients, defined in terms of barycentric and trilinear coordinates, are extended to products $P \cdot \Gamma$ and quotients Γ/P, where P is a point and Γ is a curve. In trilinears, for example, if Γ_0 denotes the circumcircle, then $P \cdot \Gamma_0$ is a parabola if and only if P lies on the Steiner inscribed ellipse. Barycentric division by the triangle center X_{110} carries Γ_0 onto the Kiepert hyperbola Γ'; if P is on Γ_0, then the point $P' = P/X_{110}$ is the point, other than the Tarry point, X_{98}, in which the line PX_{98} meets Γ', and if Ω_1 and Ω_2 denote the Brocard points, then $|P'\Omega_1|/|P'\Omega_2| = |P\Omega_1|/|P\Omega_2|$; that is, P' and P lie on the same Apollonian circle with respect to Ω_1 and Ω_2.

1. Introduction

Paul Yiu [7] gives a magnificent construction for a product $P \cdot Q$ of points in the plane of triangle ABC. If

$$P = \alpha_1 : \beta_1 : \gamma_1 \text{ and } Q = \alpha_2 : \beta_2 : \gamma_2 \tag{1}$$

are representations in homogeneous barycentric coordinates, then the Yiu product is given by

$$P \cdot Q = \alpha_1\alpha_2 : \beta_1\beta_2 : \gamma_1\gamma_2 \tag{2}$$

whenever $\{\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2\} \neq \{0\}$.

Cyril Parry [3] constructs an analogous product using trilinear coordinates. In view of the applicability of both the Yiu and Parry products, the notation in equations (1) and (2) will represent general homogeneous coordinates, as in [6, Chapter 1], unless otherwise noted. We also define the quotient

$$P/Q := \alpha_1\beta_2\gamma_2 : \beta_1\gamma_2\alpha_2 : \gamma_1\alpha_2\beta_2$$

whenever $Q \notin \{A, B, C\}$. Specialization of coordinates will be communicated by phrases such as those indicated here:

$$\left\{ \begin{matrix} \text{barycentric} \\ \text{trilinear} \end{matrix} \right\} \left\{ \begin{matrix} \text{multiplication} \\ \text{product} \\ \text{division} \\ \text{quotient} \end{matrix} \right\}.$$

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If S is a set of points, then $P \cdot S := \{P \cdot Q : \ Q \in S\}$. In particular, if S is a curve Γ, then $P \cdot \Gamma$ and Γ/P are curves, except for degenerate cases, such as when $P \in \{A, B, C\}$.

In all that follows, suppose $P = p : q : r$ is a point not on a sideline of triangle ABC, so that $pqr \neq 0$, and consequently, $U/P = \frac{u}{p} : \frac{v}{q} : \frac{w}{r}$ for all $U = u : v : w$.

Example 1. If Γ is a line $\ell\alpha + m\beta + n\gamma = 0$, then $P \cdot \Gamma$ is the line $(\ell/p)\alpha + (m/q)\beta + (n/r)\gamma = 0$ and Γ/P is the line $p\ell\alpha + qm\beta + rn\gamma = 0$. Given the line QR of points Q and R, it is easy to check that $P \cdot QR$ is the line of $P \cdot Q$ and $P \cdot R$. In particular, $P \cdot \triangle ABC = \triangle ABC$, and if T is a cevian triangle, then $P \cdot T$ is a cevian triangle.

2. Conics and Cubics

Each conic Γ in the plane of triangle ABC is given by an equation of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0. \tag{3}$$

That $P \cdot \Gamma$ is the conic

$$(u/p^2)\alpha^2 + (v/q^2)\beta^2 + (w/r^2)\gamma^2 + 2(f/qr)\beta\gamma + 2(g/rp)\gamma\alpha + 2(h/pq)\alpha\beta = 0 \tag{4}$$

is clear, since $\alpha : \beta : \gamma$ satifies (3) if and only if $p\alpha : q\beta : r\gamma$ satisfies (4). In the case of a circumconic Γ given in general form by

$$\frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} = 0, \tag{5}$$

the product $P \cdot \Gamma$ is the circumconic

$$\frac{pf}{\alpha} + \frac{qg}{\beta} + \frac{rh}{\gamma} = 0.$$

Thus, if X is the point such that $X \cdot \Gamma$ is a given circumconic $\frac{u}{\alpha} + \frac{v}{\beta} + \frac{w}{\gamma} = 0$, then $X = \frac{u}{f} : \frac{v}{g} : \frac{w}{h}$.

Example 2. In trilinears, the circumconic Γ in (5) is the isogonal transform of the line L given by $f\alpha + g\beta + h\gamma = 0$. The isogonal transform of $P \cdot L$ is Γ/P.

Example 3. Let $U = u : v : w$. The conic $W(U)$ given in [1, p. 238] by

$$u^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 - 2vw\beta\gamma - 2wu\gamma\alpha - 2uv\alpha\beta = 0$$

is inscribed in triangle ABC. The conic $P \cdot W(U)$ given by

$$(u/p)^2\alpha^2 + (v/q)^2\beta^2 + (w/r)^2\gamma^2 - 2(vw/qr)\beta\gamma - 2(wu/rp)\gamma\alpha - 2(uv/pq)\alpha\beta = 0$$

is the inscribed conic $W(U/P)$. In trilinears, we start with Γ = incircle, given by

$$u = u(a, b, c) = a(b + c - a), v = u(b, c, a), w = u(c, a, b),$$

and find [1]

[1]The conics in Example 3 are discussed in [1, p.238] as examples of a type denoted by $W(X_i)$, including incircle = $W(X_{55})$, Steiner inscribed ellipse = $W(X_6)$, Kiepert parabola = $W(X_{512})$, and Yff parabola = $W(X_{647})$. A list of X_i including trilinears, barycentrics, and remarks is given in [2].

Conic	Trilinear product	Barycentric product
Steiner inscribed ellipse	$X_9 \cdot \Gamma$	$X_8 \cdot \Gamma$
Kiepert parabola	$X_{643} \cdot \Gamma$	$X_{645} \cdot \Gamma$
Yff parabola	$X_{644} \cdot \Gamma$	$X_{646} \cdot \Gamma$

Example 4. Here we combine notions from Examples 1-3. The circumcircle, Γ_0, may be regarded as a special circumconic, and every circumconic has the form $P \cdot \Gamma_0$. We ask for the locus of a point P for which the circumconic $P \cdot \Gamma_0$ is a parabola. As such a conic is the isogonal transform of a line tangent to Γ_0, we begin with this statement of the problem: find $P = p : q : r$ (trilinears) for which the line L given by $\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma} = 0$ meets Γ_0, given by $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0$ in exactly one point. Eliminating γ leads to

$$\frac{\alpha}{\beta} = \frac{cr - ap - bq \pm \sqrt{(ap + bq - cr)^2 - 4abpq}}{2bp}.$$

We write the discriminant as

$$\Phi(p, q, r) = a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr - 2carp - 2abpq.$$

In view of Example 3 and [5, p.81], we conclude that if $W(X_6)$ denotes the Steiner inscribed ellipse, with trilinear equation $\Phi(\alpha, \beta, \gamma) = 0$, then

$$P \cdot \Gamma_0 \text{ is a } \left\{ \begin{array}{c} \text{hyperbola} \\ \text{parabola} \\ \text{ellipse} \end{array} \right\} \text{ according as } P \text{ lies } \left\{ \begin{array}{rl} \text{inside} & W(X_6) \\ \text{on} & W(X_6) \\ \text{outside} & W(X_6) \end{array} \right\}. \tag{6}$$

Returning to the case that L is tangent to Γ_0, it is easy to check that the point of tangency is $(X_1/P)©X_6$. (See Example 7 for Ceva conjugacy, denoted by ©.)

If the method used to obtain statement (6) is applied to barycentric multiplication, then a similar conclusion is reached, in which the role of $W(X_6)$ is replaced by the inscribed conic whose barycentric equation is

$$\alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta = 0,$$

that is, the ellipse $W(X_2)$.

Example 5. Suppose points P and Q are given in trilinears: $P = p : q : r$, and $U = u : v : w$. We shall find the locus of a point $X = \alpha : \beta : \gamma$ such that $P \cdot X$ lies on the line UX. This on-lying is equivalent to the determinant equation

$$\begin{vmatrix} u & v & w \\ \alpha & \beta & \gamma \\ p\alpha & q\beta & r\gamma \end{vmatrix} = 0,$$

expressible as a circumconic:

$$\frac{u(q - r)}{\alpha} + \frac{v(r - p)}{\beta} + \frac{w(p - q)}{\gamma} = 0. \tag{7}$$

One may start with the line X_1P, form its isogonal transform Γ, and then recognize (7) as $U \cdot \Gamma$. For example, in trilinears, equation (7) represents the hyperbolas of

Kiepert, Jerabek, and Feuerbach according as $(P,U) = (X_{31}, X_{75})$, (X_6, X_{48}), and (X_1, X_3); or, in barycentrics, according as $(P,U) = (X_6, X_{76})$, (X_1, X_3), and (X_2, X_{63}).

Example 6. Again in trilinears, let Γ be the self-isogonal cubic $Z(U)$ given in [1, p. 240] by

$$u\alpha(\beta^2-\gamma^2)+v\beta(\gamma^2-\alpha^2)+w\gamma(\alpha^2-\beta^2)=0.$$

This is the locus of points X such that X, X_1/X, and U are collinear; the point U is called the *pivot* of $Z(U)$. The quotient Γ/P is the cubic

$$up\alpha(q^2\beta^2-r^2\gamma^2)+vq\beta(r^2\gamma^2-p^2\alpha^2)+wr\gamma(p^2\alpha^2-q^2\beta^2)=0.$$

Although Γ/P is not generally self-isogonal, it is self-conjugate under the P^2-isoconjugacy defined (e.g., [4]) by $X \to X_1/(X\cdot P^2)$.

Example 7. Let $X©P$ denote the X-Ceva conjugate of P, defined in [1, p.57] for $X = x:y:z$ and $P = p:q:r$ by

$$X©P = p(-\frac{p}{x}+\frac{q}{y}+\frac{r}{z}) : q(-\frac{q}{y}+\frac{r}{z}+\frac{p}{x}) : r(-\frac{r}{z}+\frac{p}{x}+\frac{q}{y}).$$

Assume that $X \neq P$. It is easy to check that the locus of a point X for which $X©P$ lies on the line XP is given by

$$\frac{\alpha}{p}(\frac{\beta^2}{q^2}-\frac{\gamma^2}{r^2})+\frac{\beta}{q}(\frac{\gamma^2}{r^2}-\frac{\alpha^2}{p^2})+\frac{\gamma}{r}(\frac{\alpha^2}{p^2}-\frac{\beta^2}{q^2})=0. \tag{8}$$

In trilinears, equation (8) represents the product $P\cdot\Gamma$ where Γ is the cubic $Z(X_1)$. The locus of X for which $P©X$ lies on XP is also the cubic (8).

3. Brocard Points and Apollonian Circles

Here we discuss some special properties of the triangle centers X_{98} (the Tarry point) and X_{110} (the focus of the Kiepert parabola). X_{98} is the point, other than A, B, C, that lies on both the circumcircle and the Kiepert hyperbola.

Let ω be the Brocard angle, given by

$$\cot\omega = \cot A + \cot B + \cot C.$$

In trilinears,

$$\begin{aligned} X_{98} &= \sec(A+\omega):\sec(B+\omega):\sec(C+\omega), \\ X_{110} &= \frac{a}{b^2-c^2}:\frac{b}{c^2-a^2}:\frac{c}{a^2-b^2}. \end{aligned}$$

Theorem. *Barycentric division by X_{110} carries the circumcircle Γ_0 onto the Kiepert hyperbola Γ'. For every point P on Γ_0, the line joining P to the Tarry point X_{98} (viz., the tangent at X_{98} if $P = X_{98}$) intersects Γ' again at $P' = P/X_{110}$. Furthermore, P/X_{110} lies on the Apollonian circle of P with respect to the two Brocard points Ω_1 and Ω_2; that is*

$$\frac{|P'\Omega_1|}{|P'\Omega_2|}=\frac{|P\Omega_1|}{|P\Omega_2|}. \tag{9}$$

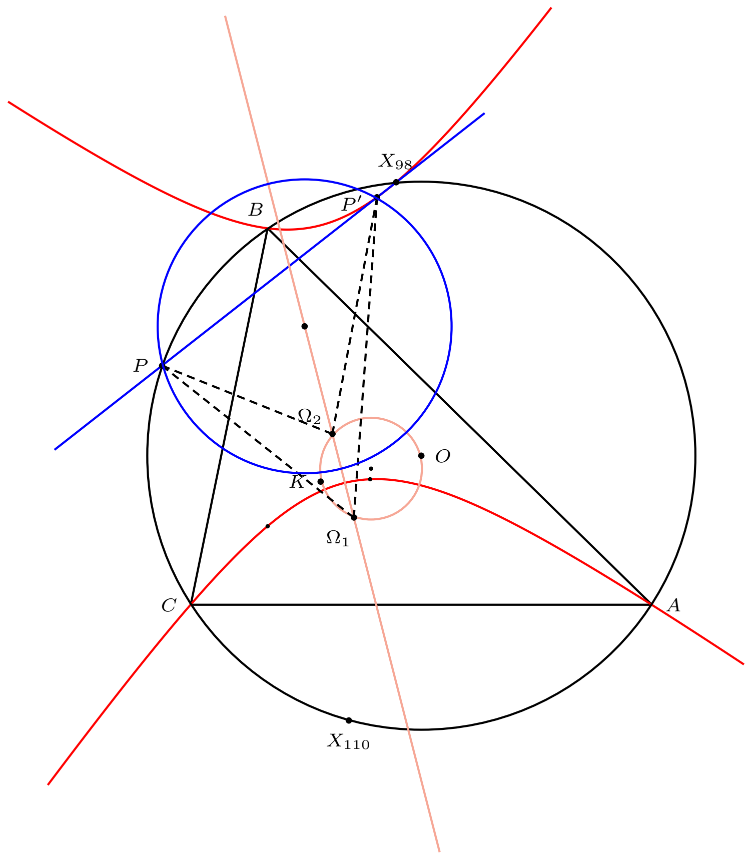

Figure 1

Proof. In barycentrics, Γ_0 and Γ' are given by

$$\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}=0 \text{ and } \frac{b^2-c^2}{\alpha}+\frac{c^2-a^2}{\beta}+\frac{a^2-b^2}{\gamma}=0,$$

and, also in barycentrics,

$$X_{110}=\frac{a}{b^2-c^2}:\frac{b}{c^2-a^2}:\frac{c}{a^2-b^2}$$

so that $\Gamma'=\Gamma_0/X_{110}$.

For the remainder of the proof, we use trilinears. A parametric representation for Γ_0 is given by

$$P=P(t)=a(1-t):bt:ct(t-1), \tag{10}$$

for $-\infty<t<\infty$, and the barycentric product P/X_{110} is given in trilinears by

$$(1-t)\frac{b^2-c^2}{a}:t\frac{c^2-a^2}{b}:t(t-1)\frac{a^2-b^2}{c}.$$

That this point lies on line PX_{98} is equivalent to the following easily verified identity:

$$\begin{vmatrix} (1-t)\frac{b^2-c^2}{a} & t\frac{c^2-a^2}{b} & t(t-1)\frac{a^2-b^2}{c} \\ a(1-t) & bt & ct(t-1) \\ \sec(A+\omega) & \sec(B+\omega) & \sec(C+\omega) \end{vmatrix} = 0.$$

We turn now to a formula [1, p.31] for the distance between two points expressed in normalized[2] trilinears (α, β, γ) and $(\alpha', \beta', \gamma')$:

$$\frac{1}{2\sigma}\sqrt{abc[a\cos A(\alpha-\alpha')^2 + b\,\cos B(\beta-\beta')^2 + c\,\cos C(\gamma-\gamma')^2]}, \qquad (11)$$

where σ denotes the area of triangle ABC. Let

$$\begin{aligned} D &= c^2t^2 - (c^2+a^2-b^2)t + a^2, \\ S &= a^2b^2 + b^2c^2 + c^2a^2. \end{aligned}$$

Normalized trilinears for (10) and the two Brocard points follow:

$$P = ((1-t)ha, thb, t(t-1)hc),$$

where $h = \frac{2\sigma}{D}$, and

$$\Omega_1 = \left(\frac{h_1c}{b}, \frac{h_1a}{c}, \frac{h_1b}{a}\right), \qquad \Omega_2 = \left(\frac{h_1b}{c}, \frac{h_1c}{a}, \frac{h_1a}{b}\right),$$

where and $h_1 = \frac{2abc\sigma}{S}$.

Abbreviate $a\cos A$, $b\cos B$, $c\cos C$, and $1-t$ as a', b', c', and t' respectively, and write

$$E = a'\left(\frac{t'ha - h_1c}{b}\right)^2 + b'\left(\frac{thb - h_1a}{c}\right)^2 + c'\left(\frac{tt'hc - h_1b}{a}\right)^2, \quad (12)$$

$$F = a'\left(\frac{t'ha - h_1b}{c}\right)^2 + b'\left(\frac{thb - h_1c}{a}\right)^2 + c'\left(\frac{tt'hc - h_1a}{b}\right)^2. \quad (13)$$

Equation (11) then gives

$$\frac{|P\Omega_1|^2}{|P\Omega_2|^2} = \frac{E}{F}. \qquad (14)$$

In (12) and (13), replace $\cos A$ by $(b^2+c^2-a^2)/2bc$, and similarly for $\cos B$ and $\cos C$, obtaining from (14) the following:

$$\frac{|P\Omega_1|^2}{|P\Omega_2|^2} = \frac{t^2a^2 - t(a^2+b^2-c^2) + b^2}{t^2b^2 - t(b^2+c^2-a^2) + c^2}.$$

[2]Sometimes trilinear coordinates are called normal coordinates. We prefer “trilinears”, so that we can say “normalized trilinears,” not “normalized normals.” One might say that the latter double usage of “normal” can be avoided by saying “actual normal distances”, but this would be unsuitable for normalization of points at infinity. Another reason for retaining “trilinear” and “quadriplanar”—not replacing both with “normal”— is that these two terms distinguish between lines and planes as the objects with respect to which normal distances are defined. In discussing points relative to a tetrahedron, for example, one could have both trilinears and quadriplanars in the same sentence.

Note that if the numerator in the last fraction is written as $f(t,a,b,c)$, then the denominator is $t^2 f(\frac{1}{t},c,b,a)$. Similarly,

$$\frac{|P'\Omega_1|^2}{|P'\Omega_2|^2} = \frac{g(t,a,b,c)}{t^4 g(\frac{1}{t},c,b,a)},$$

where

$$g(t,a,b,c) = t^4 e_4 + t^3 e_3 + t^2 e_2 + t e_1 + e_0,$$

and

$$\begin{aligned}
e_4 &= a^4 b^2 (a^2 - b^2)^2, \\
e_3 &= a^2 (a^2 - b^2)(b^6 + c^6 + 2a^2 b^2 c^2 - 2a^4 b^2 - 2a^2 c^4 - 2b^2 c^4 + a^4 c^2 + a^2 b^4), \\
e_2 &= b^2 c^2 (b^2 - c^2)^3 + a^2 c^2 (c^2 - a^2)^3 + a^2 b^2 (a^6 + 2b^6 - 3a^2 b^4) \\
&\quad + a^2 b^2 c^2 (b^4 + c^4 - 2a^4 - 4b^2 c^2 + 2a^2 c^2 + 2a^2 b^2), \\
e_1 &= b^2 (c^2 - b^2)(a^6 + c^6 - 3b^2 c^4 + 2b^4 c^2 - 2a^4 b^2 - 2a^4 c^2 + 2a^2 b^2 c^2 + a^2 b^4), \\
e_0 &= b^4 c^2 (b^2 - c^2)^2.
\end{aligned}$$

One may now verify directly, using a computer algebra system, or manually with plenty of paper, that

$$t^2 f(t,a,b,c) g(\frac{1}{t},c,b,a) = f(\frac{1}{t},c,b,a) g(t,a,b,c),$$

which is equivalent to the required equation (9). □

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