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Geometric Construction of Reciprocal Conjugations

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Abstract. Two conjugation mappings are well known in the geometry of the triangle: the isogonal and isotomic conjugations. These two are members of the family of reciprocal conjugations. In this paper we provide an easy and general construction for reciprocal conjugates of a point, given a pair of conjugate points. A connection is made to desmic configurations.

1. Introduction

Let ABC be a triangle. To represent a point in the plane of ABC we make use of homogeneous coordinates. Two such coordinate systems are well known, barycentric and normal (trilinear) coordinates. See [1] for an introduction on normal coordinates,[1] and [4] for barycentric coordinates. In the present paper we work with homogeneous barycentric coordinates exclusively.

Consider a point $X = (x : y : z)$. The isogonal conjugate X^* of X is represented by $(a^2yz : b^2xz : c^2yz)$, which we loosely write as $(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z})$ for X outside the sidelines of ABC (so that $xyz \neq 0$). In the same way the isotomic conjugate $X^\bullet$ of X is represented by $(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$. For both X^* and $X^\bullet$ the coordinates are the products of the reciprocals of those of X and the constant 'coordinates' from a certain homogeneous triple, $(a^2 : b^2 : c^2)$ and $(1 : 1 : 1)$ respectively. [2]

With this observation it is reasonable to generalize the two famous conjugations to *reciprocal conjugations*, where the homogeneous triple takes a more general form $(\ell : m : n)$ with $\ell mn \neq 0$. By the $(\ell : m : n)$-*reciprocal conjugation* or simply $(\ell : m : n)$-*conjugation*, we mean the mapping

$$\tau : (x : y : z) \mapsto \left(\frac{\ell}{x} : \frac{m}{y} : \frac{n}{z}\right).$$

It is clear that for any point X outside the side lines of ABC, $\tau(\tau(X)) = X$. A reciprocal conjugation is uniquely determined by any one pair of conjugates: if $\tau(x_0 : y_0 : z_0) = (x_1 : y_1 : z_1)$, then $\ell : m : n = x_0x_1 : y_0y_1 : z_0z_1$. It is convenient to regard $(\ell : m : n)$ as the coordinates of a point P_0, which we call the *pole* of the conjugation τ. The poles of the isogonal and isotomic conjugations,

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[1] What we call normal coordinates are traditionally called *trilinear* coordinates; they are the ratio of signed distances of the point to the side lines of the reference triangle.

[2] Analogous results hold when normal coordinates are used instead of barycentrics.

for example, are the symmedian point and the centroid respectively. In this paper we address the questions of (i) construction of τ given a pair of points P and Q conjugate under τ, (ii) construction of $\tau(P)$ given the pole R_0.

2. The parallelogram construction

2.1. *Isogonal conjugation.* There is a construction of the isogonal conjugate of a point that gives us a good opportunity for generalization to all reciprocal conjugates.

Proposition 1. *Let P be a point and let $A'B'C'$ be its pedal triangle. Let A'' be the point such that $B'PC'A''$ is a parallelogram. In the same way construct B'' and C''. Then the perspector of triangles ABC and $A''B''C''$ is the isogonal conjugate P^* of P.*

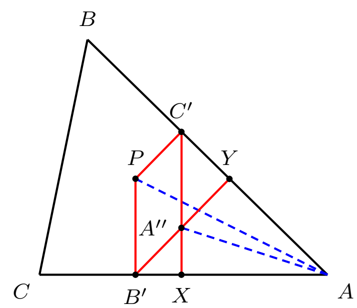

Figure 1

Proof. This is equivalent to the construction of the isogonal conjugate of P as the point of concurrency of the perpendiculars through the vertices of ABC to the corresponding sides of the pedal triangle. Here we justify it directly by noting that AP and AA'' are isogonal lines. Let $C'A''$ and $B'A''$ intersect AC and AB at X and Y respectively. Clearly,

$$B'P : YA'' = A''C' : YA'' = B'A : YA.$$

From this we conclude that triangles APB' and $AA''Y$ are similar, so that AP and AA'' are indeed isogonal lines. □

2.2. *Construction of $(\ell : m : n)$-conjugates.* Observe that the above construction depends on the 'altitudes' forming the pedal triangle $A'B'C'$. When these altitudes are replaced by segments parallel to the cevians of a point $H = (f : g : h)$, we obtain a generalization to the reciprocal conjugation

$$\tau : (x : y : z) \mapsto \left(\frac{f(g+h)}{x} : \frac{g(f+h)}{y} : \frac{h(f+g)}{z} \right).$$

In this way we get the complete set of reciprocal conjugations . In particular, given $(\ell : m : n)$, by choosing H to be the point with (homogeneous barycentric) coordinates

$$\left(\frac{1}{m+n-\ell} : \frac{1}{n+\ell-m} : \frac{1}{\ell+m-n} \right),$$

this construction gives the $(\ell : m : n)$-conjugate of points.[3]

3. The perspective triangle construction

The parallelogram construction depended on a triple of directions of cevians. These three directions can be seen as a degenerate triangle on the line at infinity, perspective to ABC. We show that this triangle can be replaced by any triangle $A_1B_1C_1$ perspective to ABC, thus making the notion of reciprocal conjugation *projective*.

Proposition 2. *A triangle $A_1B_1C_1$ perspective with ABC induces a reciprocal conjugation : for every point M not on the side lines of ABC and $A_1B_1C_1$, construct*

$$A' = A_1M \cap BC, \qquad B' = B_1M \cap CA, \qquad C' = C_1M \cap AB;$$
$$A'' = B_1C' \cap C_1B', \qquad B'' = C_1A' \cap A_1C', \qquad C'' = A_1B' \cap B_1A'.$$

Triangle $A''B''C''$ is perspective with ABC at a point N, and the correspondence $M \mapsto N$ is a reciprocal conjugation.

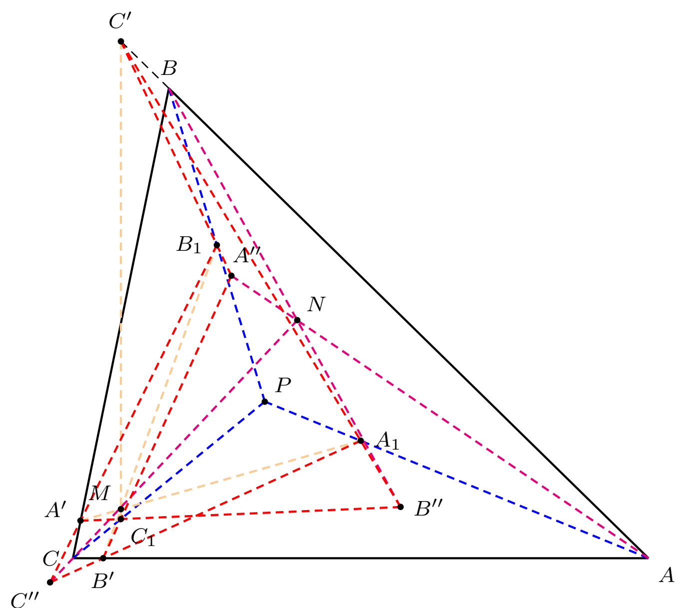

Figure 2

Proof. Since $A_1B_1C_1$ is perspective with ABC, we may write the coordinates of its vertices in the form

$$A_1 = (U : v : w), \qquad B_1 = (u : V : w), \qquad C_1 = (u : v : W). \qquad (1)$$

The perspector is $P = (u : v : w)$. Let $M = (f : g : h)$ be a point outside the sidelines of ABC and $A_1B_1C_1$. Explicitly,

$$A' = (0 : gU - fv : hU - fw) \quad \text{and} \quad B' = (fV - gu : 0 : hV - gw).$$

[3]Let X be the point with coordinates $(\ell : m : n)$. The point H can be taken as the isotomic conjugate of the point Y which divides the segment XG in the ratio $XG : GY = 1 : 2$.

The lines A_1B' and B_1A' are given by [4]

$$\begin{array}{rcl} (gw-hV)vx+(hUV-gwU-fwV+guw)y+(fV-gu)vz &=& 0, \\ (gwU-fvw-hUV+fwV)x+(hU-fw)uy+(fv-gU)uz &=& 0. \end{array}$$

These lines intersect in the point C'' with coordinates

$$(guU:fvV:fwV+gwU-hUV)\sim\left(\frac{uU}{f}:\ \frac{vV}{g}:\ \frac{fwV+gwU-hUV}{fg}\right).$$

With similar results for A'' and B'' we have the perspectivity of $A''B''C''$ and ABC at the point

$$N=\left(\frac{uU}{f}:\frac{vV}{g}:\frac{wW}{h}\right).$$

The points M and N clearly correspond to one another under the reciprocal $(uU:vV:wW)$-conjugation. □

Theorem 3. *Let P, Q, R be collinear points. Denote by X, Y, Z the traces of R on the side lines BC, CA, AB of triangle ABC, and construct triangle $A_1B_1C_1$ with vertices*

$$A_1=PA\cap QX,\quad B_1=PB\cap QY,\quad C_1=PC\cap QZ. \tag{2}$$

Triangle $A_1B_1C_1$ is perspective with ABC, and induces the reciprocal conjugation under which P and Q correspond.

Proof. If $P=(u:v:w)$, $Q=(U:V:W)$, and $R=(1-t)P+tQ$ for some $t\neq 0,1$, then

$$A_1=\left(\frac{-tU}{1-t}:v:w\right),\qquad B_1=\left(u:\frac{-tV}{1-t}:w\right),\qquad C_1=\left(u:v:\frac{-tW}{1-t}\right).$$

The result now follows from Proposition 2 and its proof. □

Proposition 2 and Theorem 3 together furnish a construction of $\tau(M)$ for an arbitrary point M (outside the side lines of ABC) under the conjugation τ defined by two distinct points P and Q. In particular, the pole P_0 can be constructed by applying to the triangle $A_1B_1C_1$ in (2) and M the centroid of ABC in the construction of Proposition 2.

Corollary 4. *Let P_0 be a point different from the centroid G of triangle ABC, regarded as the pole of a reciprocal conjugation τ. To construct $\tau(M)$, apply the construction in Theorem 3 to $(P,Q)=(G,P_0)$. The choice of R can be arbitrary, say, the midpoint of GP_0.*

Remark. This construction does not apply to isotomic conjugation, for which the pole is the centroid.

[4] Here we have made use of the fact that the line B_1C_1 is given by the equation

$$w(v-V)x+w(u-U)y+(UV-uv)z=0,$$

so that we indeed can divide by $fw(v-V)+gw(u-U)+h(UV-uv)$.

4. Desmic configuration

We take a closer look of the construction in Proposition 2. Given triangle $A_1B_1C_1$ with perspector P, it is known that the triangle $A_2B_2C_2$ with vertices

$$A_2 = BC_1 \cap CB_1, \qquad B_2 = AC_1 \cap CA_1, \qquad C_2 = AB_1 \cap BA_1,$$

is perspective to both ABC and $A_1B_1C_1$, say at points Q and R respectively, and that the perspectors P, Q, R are collinear. See, for example, [2]. Indeed, if the vertices of $A_1B_1C_1$ have coordinates given by (1), those of $A_2B_2C_2$ have coordinates

$$A_2 = (u : V : W), \quad B_2 = (U : v : W), \qquad C_2 = (U : V : w).$$

From these, it is clear that

$$Q = (U : V : W) \qquad \text{and} \qquad R = (u + U : v + V : w + W).$$

Triangle $A_2B_2C_2$ is called the *desmic mate* of triangle $A_1B_1C_1$. The three triangles, their perspectors, and the connecting lines form a *desmic configuration, i.e.*, each line contains 3 points and each point is contained in 4 lines. This configuration also contains the three desmic quadrangles: $ABCR$, $A_1B_1C_1Q$ and $A_2B_2C_2P$.

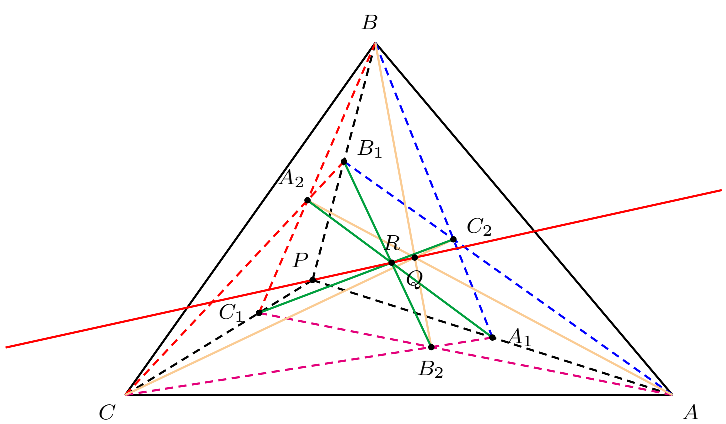

Figure 3

The construction in the preceding section shows that given collinear points P, Q, and R, there is a desmic configuration as above in which the quadrangles $A_1B_1C_1Q$ and $A_2B_2C_2P$ are perspective at R. The reciprocal conjugations induced by $A_1B_1C_1$ and $A_2B_2C_2$ are the same, and is independent of the choice of R.

Barry Wolk [3] has observed that these twelve points all lie on the *iso*-$(uU : vV : wW)$ cubic with pivot R: [5]

$$x(u+U)(wWy^2 - vVz^2) + y(v+V)(uUz^2 - wWx^2) + z(w+W)(vVx^2 - uUy^2) = 0.$$

[5] An iso-$(\ell : m : n)$ cubic with pivot R is the locus of points X for which X and its iso-$(\ell : m : n)$-conjugate are collinear with R.

Since reciprocal conjugates link the vertices of ABC to their opposite sides, clearly the traces of R are also on the cubic. By the symmetry of the desmic configurations, the traces of Q in $A_1B_1C_1$ and P in $A_2B_2C_2$ are also on the desmic cubic.

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