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Euler's Formula and Poncelet's Porism

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1. Introduction

It is well known [2, p. 187] that two intersecting circles $O(R)$ and $O_1(R_1)$ are the circumcircle and an excircle respectively of a triangle if and only if the Euler formula

$$d^2 = R^2 + 2RR_1, \tag{1}$$

where $d = |OO_1|$, holds. We present a possibly new proof and an application to the Poncelet porism.

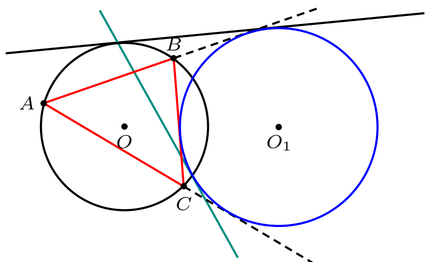

Figure 1

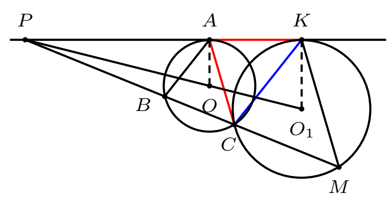

Figure 2

Theorem 1. *Intersecting circles (O) and (O_1) are the circumcircle and an excircle of a triangle if and only if the tangent to (O_1) at an intersection of the circles meets (O) again at the touch point of a common tangent.*

Proof. (Sufficiency) Let $O(R)$ and $O_1(R_1)$ be intersecting circles. (These circles are not assumed to be related to a triangle as in Figure 1.) Of the two lines tangent to both circles, let AK be one of them, as in Figure 2. Let $P = AK \cap OO_1$. Of the two points of intersection of (O) and (O_1), let C be the one not on the same side of line OO_1 as point A. Line AC is tangent to circle $O_1(R_1)$ if and only if $|AC| = |AK|$. Let B and M be the points other than C where line PC meets circles $O(R)$ and $O_1(R_1)$, respectively. Triangles ABC and KCM are homothetic with ratio $\frac{R}{R_1}$, so that $\dfrac{|AB|}{|CK|} = \dfrac{R}{R_1}$. Also, triangles ABC and CAK are similar, since $\angle ABC = \angle CAK$ and $\angle BAC = \angle ACK$. Therefore, $\dfrac{|AB|}{|AC|} = \dfrac{|AC|}{|CK|}$, so that $\dfrac{|CK|}{|AC|} \cdot \dfrac{R}{R_1} = \dfrac{|AC|}{|CK|}$, and

$$|CK| = |AC|\sqrt{\frac{R_1}{R}}. \tag{2}$$

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Also,

$$\begin{aligned} |AK| &= |AC|\cos(\angle CAK) + |CK|\cos(\angle CKA) \\ &= |AC|\sqrt{1-\frac{|AC|^2}{4R^2}} + |CK|\sqrt{1-\frac{|CK|^2}{4R_1^2}}. \end{aligned} \tag{3}$$

If $|AC| = |AK|$, then equations (2) and (3) imply

$$|AK| = |AK|\sqrt{1-\frac{|AK|^2}{4R^2}} + |AK|\sqrt{\frac{R_1}{R}-\frac{|AK|^2}{4R^2}},$$

which simplifies to $|AK|^2 = 4RR_1 - R_1^2$. Since $|AK|^2 = d^2 - (R-R_1)^2$, where $d = |OO_1|$, we have the Euler formula given in (1). □

We shall prove the converse below from Poncelet's porism.

2. Poncelet porism

Suppose triangle ABC has circumcircle $O(R)$ and incircle $I(r)$. The Poncelet porism is the problem of finding all triangles having the same circumcircle and incircle, and the well known solution is an infinite family of triangles. Unless triangle ABC is equilateral, these triangles vary in shape, but even so, they may be regarded as "rotating" about a fixed incircle and within a fixed circumcircle.

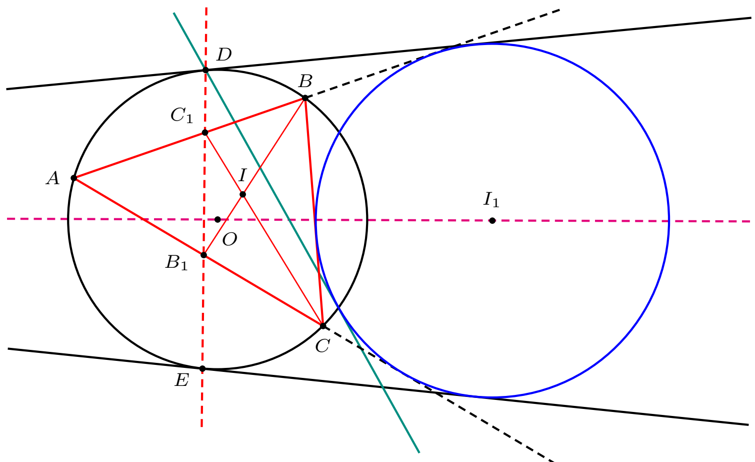

Figure 3

Continuing with the proof of the necessity part of Theorem 1, let $I_1(r_1)$ be the excircle corresponding to vertex A. Since Euler's formula holds for this configuration, the conditions for the Poncelet porism (e.g. [2, pp. 187-188]) hold. In the family of rotating triangles ABC there is one whose vertices A and B coincide in a point, D, and the limiting line AB is, in this case, tangent to the excircle. Moreover, lines CA and BC coincide as the line tangent to the excircle at a point of intersection of the circles, as in Figure 3. This completes the proof of Theorem 1.

Certain points of triangle ABC, other than the centers of the two fixed circles, stay fixed during rotation ([1, p.16-19]). We can also find a fixed line in the Poncelet porism.

Theorem 2. *For each of the rotating triangles ABC with fixed circumcircle and excircle corresponding to vertex A, the feet of bisectors BB_1 and CC_1 traverse line DE, where E is the touch point of the second common tangent.*

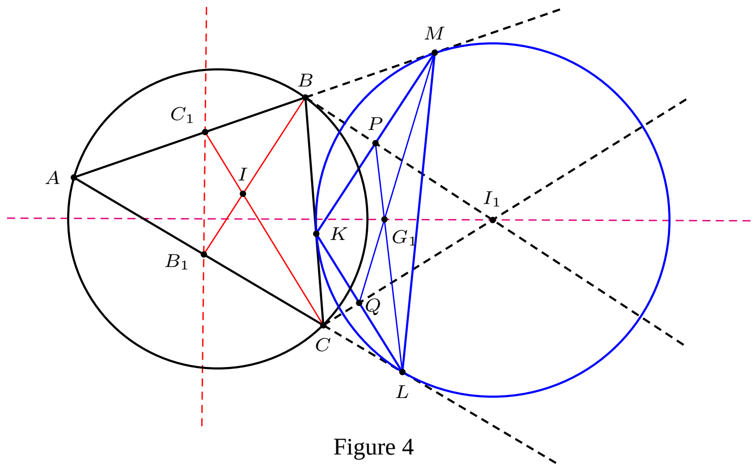

Figure 4

3. Proof of Theorem 2

We begin with the pole-polar correspondence between points and lines for the excircle with center I_1, as in Figure 4.

The polars of A, B, C are LM, MK, KL, respectively, where ΔKLM is the A-extouch triangle. As BB_1 is the internal bisector of angle B and BI_1 is the external bisector, we have $BB_1 \perp BI_1$, and the pole of BB_1 lies on the polar of B, namely MK. Therefore the pole of BB_1 is the midpoint P of segment MK. Similarly, the pole of the bisector CC_1 is the midpoint Q of segment KL. The polar of B_1 is the line passing through the poles of BB_1 and LB_1, i.e. line PL. Likewise, MQ is the polar of C_1, and the pole of B_1C_1 is centroid of triangle KLM, which we denote as G_1.

We shall prove that G_1 is fixed by proving that the orthocenter H_1 of triangle KLM is fixed. (Gallatly [1] proves that the orthocenter of the intouch triangle stays fixed in the Poncelet porism with fixed circumcircle and incircle; we offer a different proof, which applies also to the circumcircle and an excircle.)

Lemma 3. *The orthocenter H_1 of triangle KLM stays fixed as triangle ABC rotates.*

Proof. Let KLM be the extouch triangle of triangle ABC, let RST be the orthic triangle of triangle KLM, and let H_1 and E_1 be the orthocenter and nine-point center, respectively, of triangle KLM, as in Figure 5.

(1) The circumcircle of triangle RST is the nine-point circle of triangle KLM, so that its radius is equal to $\frac{1}{2}R_1$, and its center E_1 is on the Euler line I_1H_1 of triangle KLM.

(2) It is known that altitudes of an obtuse triangle are bisectors (one internal and two external) of its orthic triangle, so that H_1 is the R-excenter of triangle RST.

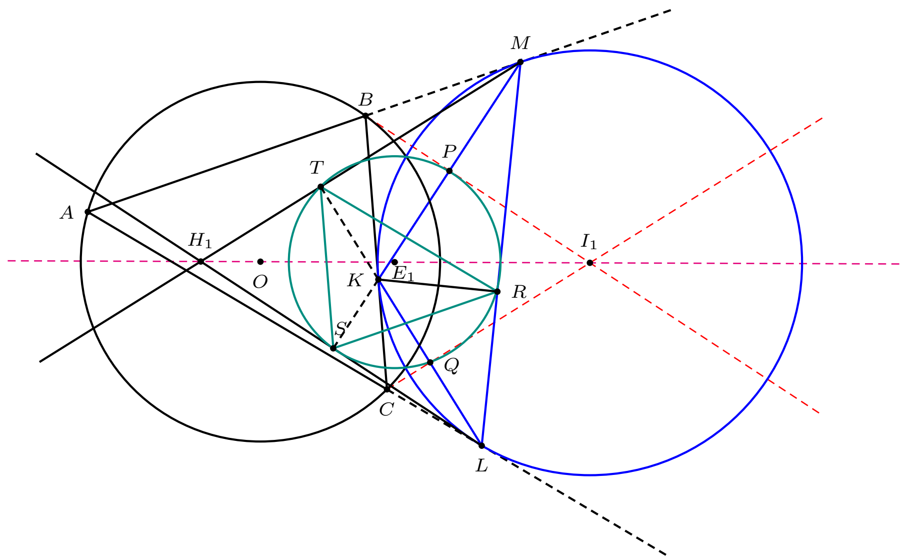

Figure 5

(3) Triangle RST and triangle ABC are homothetic. To see, for example, that $AB||RS$, we have $\angle KRL = \angle KSL = 90°$, so that L, R, S, K are concyclic. Thus, $\angle KLR = \angle KSR = \angle RSM$. On the other hand, $\angle KLR = \angle KLM = \angle KMB$ and $\angle RSM = \angle SMB$. Consequently, $AB||RS$.

(4) The ratio k of homothety of triangle ABC and triangle RST is equal to the ratio of their circumradii, i.e. $k = \frac{2R}{R_1}$. Under this homothety, $O \to E_1$ (the circumcenters) and $I_1 \to H_1$ (the excenter). It follows that $OI_1||E_1H_1$. Since E_1, I_1, H_1 are collinear, O, I_1, H_1 are collinear. Thus OI_1 is the fixed Euler line of every triangle KLM.

The place of H stays fixed on OI because $EH = \frac{OI}{k}$ remains constant. Therefore the centroid of, triangle KLM also stays fixed. □

To complete the proof of Theorem 2, note that by Lemma 3, G_1 is fixed on line OI_1. Therefore, line B_1C_1, as the polar of G_1, is fixed. Moreover, $B_1C_1 \perp OI_1$. Considering the degenerate case of the Poncelet porism, we conclude that B_1C_1 coincides with DE, as in Figure 3.

References

[1] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd edition, Francis Hodgson, London, 1913.
[2] R. A. Johnson, *Modern Geometry*, Houghton Mifflin, Boston, 1929.

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