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Conics Associated with a Cevian Nest

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Abstract. Various mappings in the plane of $\triangle ABC$ are defined in the context of a cevian nest consisting of $\triangle ABC$, a cevian triangle, and an anticevian triangle. These mappings arise as Ceva conjugates, cross conjugates, and cevapoints. Images of lines under these mappings and others, involving trilinear and conic-based poles and polars, include certain conics that are the focus of this article.

1. Introduction

Suppose L is a line in the plane of $\triangle ABC$, but not a sideline BC, CA, AB, and suppose a variable point Q traverses L. The isogonal conjugate of Q traces a conic called the isogonal transform of L, which, as is well known, passes through the vertices A, B, C. In this paper, we shall see that for various other transformations, the transform of L is a conic. These include Ceva and cross conjugacies, cevapoints, and pole-to-pole mappings[1]. Let

$$P = p_1 : p_2 : p_3 \tag{1}$$

be a point[2] not on a sideline of $\triangle ABC$. Suppose

$$U = u_1 : u_2 : u_3 \quad \text{and} \quad V = v_1 : v_2 : v_3 \tag{2}$$

are distinct points on L. Then L is given parametrically by

$$Q_t = u_1 + v_1 t : u_2 + v_2 t : u_3 + v_3 t, \; -\infty < t \leq \infty, \tag{3}$$

where $Q_\infty := V$. The curves in question can now be represented by the form $P * Q_t$ (or $P_t * Q$), where $*$ represents any of the various mappings to be considered. For any such curve, a parametric representation is given by the form

$$x_1(t) : x_2(t) : x_3(t),$$

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[1]The cevian triangle of a point P not on a sideline of ABC is the triangle $A'B'C'$, where $A' = PA \cap BC$, $B' = PB \cap CA$, $CB' = PC \cap AB$. The name *cevian* (pronounced cheh´vian) honors Giovanni Ceva (pronounced Chay´va). We use a lower case c in adjectives such as *anticevian* (cf. *nonabelian*) and a capital when the name stands alone, as in *Ceva conjugate.* The name *anticevian* derives from a special case called the *anticomplementary triangle*, so named because its vertices are the anticomplements of A, B, C.

[2]Throughout, coordinates for points are homogeneous trilinear coordinates.

where the coordinates are polynomials in t having no common nonconstant polynomial factor. The degree of the curve is the maximum of the degrees of the polynomials. When this degree is 2, the curve is a conic, and the following theorem (e.g. [5, pp. 60–65]) applies.

Theorem 1. *Suppose a point $X = x_1 : x_2 : x_3$ is given parametrically by*

$$\begin{aligned} x_1 &= d_1t^2 + e_1t + f_1 && (4)\\ x_2 &= d_2t^2 + e_2t + f_2 && (5)\\ x_3 &= d_3t^2 + e_3t + f_3, && (6) \end{aligned}$$

where the matrix

$$M = \begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}$$

is nonsingular with adjoint (cofactor) matrix

$$M^{\#} = \begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Then X lies on the conic:

$$(E_1\alpha + E_2\beta + E_3\gamma)^2 = (D_1\alpha + D_2\beta + D_3\gamma)(F_1\alpha + F_2\beta + F_3\gamma). \tag{7}$$

Proof. Since M is nonsingular, its determinant δ is nonzero, and $M^{-1} = \frac{1}{\delta}M^{\#}$. Let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix},$$

so that $X = MT$ and $M^{-1}X = T$. This second equation is equivalent to the system

$$\begin{array}{ccccccc} D_1x_1 & + & D_2x_2 & + & D_3x_3 & = & \delta t^2 \\ E_1x_1 & + & E_2x_2 & + & E_3x_3 & = & \delta t \\ F_1x_1 & + & F_2x_2 & + & F_3x_3 & = & \delta. \end{array}$$

The equal quotients $\delta t^2/\delta t$ and $\delta t/\delta$ yield

$$\frac{D_1x_1 + D_2x_2 + D_3x_3}{E_1x_1 + E_2x_2 + E_3x_3} = \frac{E_1x_1 + E_2x_2 + E_3x_3}{F_1x_1 + F_2x_2 + F_3x_3}.$$

□

For a first example, suppose $Q = q_1 : q_2 : q_3$ is a point not on a sideline of $\triangle ABC$, and let L be the line $q_1\alpha + q_2\beta + q_3\gamma = 0$. The P-isoconjugate of Q, is (e.g., [4, Glossary]) the point

$$P * Q = \frac{1}{p_1q_1} : \frac{1}{p_2q_2} : \frac{1}{p_3q_3}.$$

The method of proof of Theorem 1 shows that the P-isoconjugate of L (i.e., the set of points $P * R$ for R on L) is the circumconic

$$\frac{q_1}{p_1\alpha} + \frac{q_2}{p_2\beta} + \frac{q_3}{p_3\gamma} = 0.$$

We shall see that the same method applies to many other configurations.

2. Cevian nests and two conjugacies

A fruitful configuration in the plane of $\triangle ABC$ is the cevian nest, consisting of three triangles T_1, T_2, T_3 such that T_2 is a cevian triangle of T_1, and T_3 is a cevian triangle of T_2. In this article, $T_2 = \triangle ABC$, so that T_1 is the anticevian triangle of some point P, and T_3 is the cevian triangle of some point Q. It is well known (e.g. [1, p.165]) that if any two pairs of such triangles are perspective pairs, then the third pair are perspective also[3]. Accordingly, for a cevian nest, given two of the perspectors, the third may be regarded as the value of a binary operation applied to the given perspectors. There are three such pairs, hence three binary operations. As has been noted elsewhere ([2, p. 203] and [3, Glossary]), two of them are involutory: Ceva conjugates and cross conjugates.

2.1. *Ceva conjugate.* The *P-Ceva conjugate of Q*, denoted by $P©Q$, is the perspector of the cevian triangle of P and the anticevian triangle of Q; for $P = p_1 : p_2 : p_3$ and $Q = q_1 : q_2 : q_3$, we have

$$P©Q = q_1(-\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}) : q_2(\frac{q_1}{p_1} - \frac{q_2}{p_2} + \frac{q_3}{p_3}) : q_3(\frac{q_1}{p_1} + \frac{q_2}{p_2} - \frac{q_3}{p_3}).$$

Theorem 2. *Suppose P, U, V, Q_t are points as in (1)-(3); that is, Q_t traverses line UV. The locus of $P©Q_t$ is the conic*

$$\frac{\alpha^2}{p_1q_1} + \frac{\beta^2}{p_2q_2} + \frac{\gamma^2}{p_3q_3} - (\frac{1}{p_2q_3} + \frac{1}{p_3q_2})\beta\gamma - (\frac{1}{p_3q_1} + \frac{1}{p_1q_3})\gamma\alpha - (\frac{1}{p_1q_2} + \frac{1}{p_2q_1})\alpha\beta = 0, \quad (8)$$

where $Q := q_1 : q_2 : q_3$, the trilinear pole of the line UV, is given by

$$q_1 : q_2 : q_3 = \frac{1}{u_2v_3 - u_3v_2} : \frac{1}{u_3v_1 - u_1v_3} : \frac{1}{u_1v_2 - u_2v_1}.$$

This conic[4] passes through the vertices of the cevian triangles of P and Q.

Proof. First, it is easy to verify that equation (8) holds for $\alpha : \beta : \gamma$ equal to any of these six vertices:

$$0 : p_2 : p_3, \quad p_1 : 0 : p_3, \quad p_1 : p_2 : 0, \quad 0 : q_2 : q_3, \quad q_1 : 0 : q_3, \quad q_1 : q_2 : 0 \qquad (9)$$

[3]Peter Yff has observed that in [1], Court apparently overlooked the fact that $\triangle ABC$ and any inscribed triangle are triply perspective, with perspectors A, B, C. For these cases, Court's result is not always true. It seems that he intended his inscribed triangles to be cevian triangles.

[4]The general equation (8) for the circumconic of two cevian triangles is one of many interesting equations in Peter Yff's notebooks.

A conic is determined by any five of its points, so it suffices to prove that the six vertices are of the form $P©Q_t$. Putting $x_1 = 0$ in (4) gives roots

$$t_a = \frac{-e_1 \pm \sqrt{e_1^2 - 4d_1 f_1}}{2d_1}, \tag{10}$$

where

$$\begin{aligned} d_1 &= v_1\left(-\frac{v_1}{p_1} + \frac{v_2}{p_2} + \frac{v_3}{p_3}\right), & (11)\\ e_1 &= -\frac{2u_1 v_1}{p_1} + \frac{u_1 v_2 + u_2 v_1}{p_2} + \frac{u_1 v_3 + u_3 v_1}{p_3}, & (12)\\ f_1 &= u_1\left(-\frac{u_1}{p_1} + \frac{u_2}{p_2} + \frac{u_3}{p_3}\right). & (13) \end{aligned}$$

The discriminant in (10) is a square, and t_a simplifies:

$$t_a = \frac{-e_1 p_2 p_3 q_2 q_3 \pm (p_3 q_2 - p_2 q_3)}{2d_1 p_2 p_3 q_2 q_3}.$$

If the numerator is $-e_1 p_2 p_3 q_2 q_3 + (p_3 q_2 - p_2 q_3)$, then (5) and (6), and substitutions for $d_2, e_2, f_2, d_3, e_3, f_3$ obtained cyclically from (11)-(13), give $x_2/x_3 = p_2/p_3$, so that $P©Q_{t_a} = 0 : p_2 : p_3$. On the other hand, if the numerator is $-e_1 p_2 p_3 q_2 q_3 - (p_3 q_2 - p_2 q_3)$, then $x_2/x_3 = q_2/q_3$ and $P©Q_{t_a} = 0 : q_2 : q_3$. Likewise, the roots t_b and t_c of (5) and (6) yield a proof that the other four vertices in (9) are of the form $P©Q_t$. □

Corollary 2.1. *Suppose $P = p_1 : p_2 : p_3$ is a point and L given by $\ell_1\alpha + \ell_2\beta + \ell_3\gamma = 0$ is a line. Suppose the point Q_t traverses L. The locus of $P©Q_t$ is the conic*

$$\frac{\ell_1\alpha^2}{p_1} + \frac{\ell_2\beta^2}{p_2} + \frac{\ell_3\gamma^2}{p_3} - \left(\frac{\ell_3}{p_2} + \frac{\ell_2}{p_3}\right)\beta\gamma - \left(\frac{\ell_1}{p_3} + \frac{\ell_3}{p_1}\right)\gamma\alpha - \left(\frac{\ell_2}{p_1} + \frac{\ell_1}{p_2}\right)\alpha\beta = 0. \tag{14}$$

Proof. Let U, V be distinct points on L, and apply Theorem 2. □

Corollary 2.2. *The conic (14) is inscribed to $\triangle ABC$ if and only if the line $L = UV$ is the trilinear pole of P.*

Proof. In this case, $\ell_1 : \ell_2 : \ell_3 = 1/p_1 : 1/p_2 : 1/p_3$, so that $P = Q$. The cevian triangles indicated by (9) are now identical, and the six pass-through points are three tangency points. □

One way to regard Corollary 2.2 is to start with an inscribed conic Γ. It follows from the general equation for such a conic (e.g., [2, p.238]) that the three touch points are of the form $0 : p_2 : p_3, p_1 : 0 : p_3, p_1 : p_2 : 0$, for some $P = p_1 : p_2 : p_3$. Then Γ is the locus of $P©Q_t$ as Q_t traverses L.

Example 1. Let $P =$ centroid and $Q =$ orthocenter. Then line UV is given by

$$(\cos A)\alpha + (\cos B)\beta + (\cos C)\gamma = 0,$$

and the conic (8) is the nine-point circle. The same is true for $P =$ orthocenter and $Q =$ centroid.

Example 2. Let P = orthocenter and $Q = X_{648}$, the trilinear pole of the Euler line, so that UV is the Euler line. The conic (8) passes through the vertices of the orthic triangle, and X_4, X_{113}, X_{155}, X_{193}, which are the P-Ceva conjugates of X_4, X_{30}, X_3, X_2, respectively.[5]

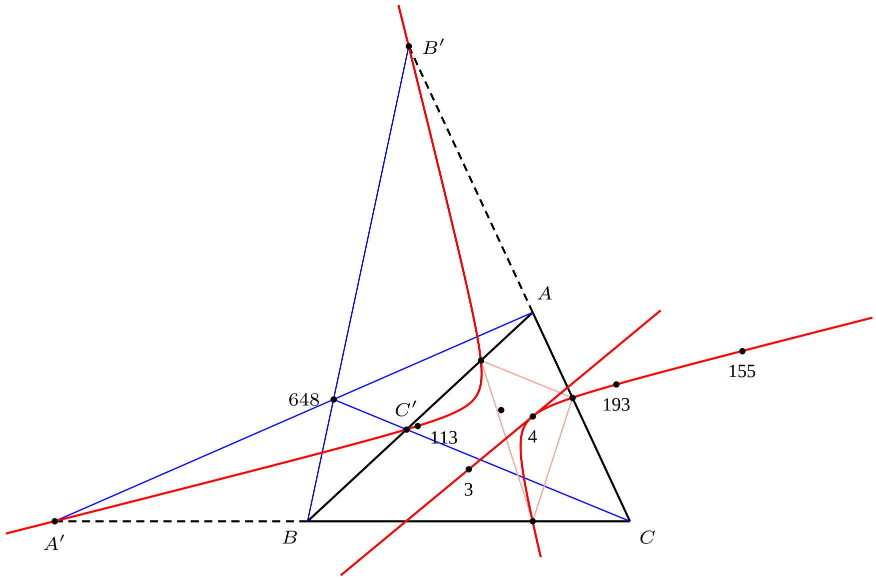

Figure 1

2.2. *Cross conjugate.* Along with Ceva conjugates, cevian nests proffer cross conjugates. Suppose $P = p_1 : p_2 : p_3$ and $Q = q_1 : q_2 : q_3$ are distinct points, neither lying on a sideline of $\triangle ABC$. Let $A'B'C'$ be the cevian triangle of Q. Let

$$A'' = PA' \cap B'C', \quad B'' = PB' \cap C'A', \quad C'' = PC' \cap A'B',$$

so that $A''B''C''$ is the cevian triangle (in $\triangle A'B'C'$) of P. The *cross conjugate* $P \otimes Q$ is the perspector of $\triangle ABC$ and $\triangle A''B''C''$. It has coordinates

$$\frac{q_1}{-\frac{p_1}{q_1}+\frac{p_2}{q_2}+\frac{p_3}{q_3}} : \frac{q_2}{-\frac{p_2}{q_2}+\frac{p_3}{q_3}+\frac{p_1}{q_1}} : \frac{q_3}{-\frac{p_3}{q_3}+\frac{p_1}{q_1}+\frac{p_2}{q_2}}.$$

It is easy to verify directly that $\otimes$ is a conjugacy; i.e., $P \otimes (P \otimes Q) = Q$, or to reach the same conclusion using the identity

$$X \otimes P = (X^{-1} \copyright P^{-1})^{-1},$$

where $(\ )^{-1}$ signifies isogonal conjugation.

The locus of $P \otimes Q_t$ is generally a curve of degree 5. However, on switching the roles of P and Q, we obtain a conic, as in Theorem 3. Specifically, let $Q = q_1 : q_2 : q_3$ remain fixed while

$$P_t = u_1 + v_1 t : u_2 + v_2 t : u_3 + v_3 t, \ -\infty < t \le \infty,$$

ranges through the line UV.

[5]Indexing of triangle centers is as in [3].

Theorem 3. *The locus of the* $P_t \otimes Q$ *is the circumconic*

$$(\frac{p_3}{q_2}+\frac{p_2}{q_3})\beta\gamma+(\frac{p_1}{q_3}+\frac{p_3}{q_1})\gamma\alpha+(\frac{p_2}{q_1}+\frac{p_1}{q_2})\alpha\beta=0, \tag{15}$$

where line UV *is represented as*

$$p_1\alpha+p_2\beta+p_3\gamma=(u_2v_3-u_3v_2)\alpha+(u_3v_1-u_1v_3)\beta+(u_1v_2-u_2v_1)\gamma=0.$$

Proof. Following the proof of Theorem 1, let

$$u_1'=-\frac{u_1}{q_1}+\frac{u_2}{q_2}+\frac{u_3}{q_3},\ v_1'=-\frac{v_1}{q_1}+\frac{v_2}{q_2}+\frac{v_3}{q_3},$$

and similarly for u_2', u_3', v_2', v_3'. Then

$$d_1=q_1v_2'v_3',\quad e_1=q_1(u_2'v_3'+u_3'v_2'),\quad f_1=q_1u_2'u_3',$$

and similarly for $d_i,\ e_i,\ f_i,\ i=2,3$. The nine terms $d_i,\ e_i,\ f_i$, yield the nine cofactors D_i, E_i, F_i, which then yield 0 for the coefficients of $\alpha^2, \beta^2, \gamma^2$ in (7) and the other three coefficients as asserted in (15). □

Example 3. Regarding the conic (15), suppose $P=p_1:p_2:p_3$ is an arbitrary triangle center and Γ is an arbitrary circumconic $\ell/\alpha+m/\beta+n/\gamma=0$. Let

$$\begin{aligned}Q &= q_1:q_2:q_3\\ &= \frac{1}{p_1(-p_1\ell+p_2m+p_3n)}:\frac{1}{p_2(-p_2m+p_3n+p_1\ell)}:\frac{1}{p_3(-p_3n+p_1\ell+p_2m)}.\end{aligned}$$

For P_t ranging through the line L given by $p_1\alpha+p_2\beta+p_3\gamma=0$, the locus of $P_t\otimes Q$ is then Γ, since

$$\frac{p_3}{q_2}+\frac{p_2}{q_3}:\frac{p_1}{q_3}+\frac{p_3}{q_1}:\frac{p_2}{q_1}+\frac{p_1}{q_2}=\ell:m:n.$$

In other words, given P and L, there exists Q such that $P_t\otimes Q$ ranges through any prescribed circumconic. In fact, Q is the isogonal conjugate of $P©L'$, where L' denotes the pole of line L. Specific cases are summarized in the following table.

P	Q	ℓ	pass-through points, X_i, for $i=$
X_1	X_1	1	$88, 100, 162, 190$ (Steiner ellipse)
X_1	X_2	$b+c$	$80, 100, 291$ (ellipse)
X_1	X_6	$a(b+c)$	$101, 190, 292$ (ellipse)
X_1	X_{57}	a	$74, 98, 99, ..., 111, 112,$ ...(circumcircle)
X_1	X_{63}	$\sin 2A$	$109, 162, 163, 293$ (ellipse)
X_1	X_{100}	$b-c$	$1, 2, 28, 57, 81, 88, 89, 105,$ ...(hyperbola)
X_1	X_{101}	$a(b-c)(b+c-a)$	$6, 9, 19, 55, 57, 284, 333,$ (hyperbola)
X_1	X_{190}	$a(b-c)$	$1, 6, 34, 56, 58, 86, 87, 106,$ ...(hyperbola)

3. Poles and polars

In this section, we shall see that, in addition to mappings discussed in §2, certain mappings defined in terms of poles and polars are nicely represented in terms of Ceva conjugates and cross conjugates

We begin with definitions. Suppose $A'B'C'$ is the cevian triangle of a point P not on a sideline of $\triangle ABC$. By Desargues's Theorem, the points $BC \cap B'C'$, $CA \cap C'A'$, $AB \cap A'B'$ are collinear. Their line is the *trilinear polar of* P. Starting with a line L, the steps reverse, yielding the *trilinear pole of* L. If L is given by $x\alpha + y\beta + z\gamma = 0$ then the trilinear pole of L is simply $1/x : 1/y : 1/z$.

Suppose Γ is a conic and X is a point. For each U on Γ, let V be the point other than U in which the line UX meets Γ, and let X' be the harmonic conjugate of X with respect to U and V. As U traverses Γ, the point X' traverses a line, the polar of X with respect to Γ, or Γ-*based polar of* X. Here, too, as with the trilinear case, for given line L, the steps reverse to define the Γ-*based pole of* L.

In §2, two mappings were defined in the context of a cevian nest. We return now to the cevian nest to define a third mapping. Suppose $P = p : q : r$ and $X = x : y : z$ are distinct points, neither lying on a sideline of $\triangle ABC$. Let $A''B''C''$ be the anticevian triangle of X. Let

$$A' = PA'' \cap BC, \quad B' = PB'' \cap CA, \quad C' = PC'' \cap AB.$$

The *cevapoint* of P and X is the perspector, R, of triangles ABC and $A'B'C'$. Trilinears are given by

$$R = \frac{1}{qz+ry} : \frac{1}{rx+pz} : \frac{1}{py+qx}. \tag{16}$$

It is easy to verify that $P = R \copyright X$.

The general conic Γ is given by the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2p\beta\gamma + 2q\gamma\alpha + 2r\alpha\beta = 0,$$

and the Γ-based polar of $X = x : y : z$ is given (e.g., [5]) by

$$(ux + ry + qz)\alpha + (vy + pz + rx)\beta + (wz + qx + py)\gamma = 0. \tag{17}$$

Example 4. Let Γ denote the circumconic $p/\alpha + q/\beta + r/\gamma = 0$, that is, the circumconic having as pivot the point $P = p : q : r$. The Γ-based polar of X is the trilinear polar of the cevapoint of P and X, given by

$$(qz + ry)\alpha + (rx + pz)\beta + (py + qx)\gamma = 0.$$

In view of (16), (trilinear polar of X) = (Γ-based polar of $X \copyright P$).

Example 5. Let Γ denote conic determined as in Theorem 2 by points P and Q. The conic is inscribed in $\triangle ABC$ if and only if $P = Q$, and in this case, the Γ-based polar of X is given by

$$\frac{1}{p}\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right)\alpha + \frac{1}{q}\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right)\beta + \frac{1}{r}\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right)\gamma = 0.$$

In other words, (Γ-based polar of X) $=$ (trilinear polar of $X \otimes P$). In particular, choosing $P = X_7$, we obtain the incircle-based polar of X:

$$f(A,B,C)\alpha + f(B,C,A)\beta + f(C,A,B)\gamma = 0,$$

where

$$f(A,B,C) = \frac{\sec^2\frac{A}{2}}{-x\cos^2\frac{A}{2} + y\cos^2\frac{B}{2} + z\cos^2\frac{C}{2}}.$$

Suppose now that Γ is a conic and L a line. As a point

$$X = p_1 + q_1 t : p_2 + q_2 t : p_3 + q_3 t \tag{18}$$

traverses L, a mapping is defined by the trilinear pole of the Γ-based polar of X. This pole has trilinears found directly from (17):

$$\frac{1}{g_1(t)} : \frac{1}{g_2(t)} : \frac{1}{g_3(t)},$$

where $g_1(t) = u(p_1 + q_1 t) + r(p_2 + q_2 t) + q(p_3 + q_3 t)$, and similarly for $g_2(t)$ and $g_3(t)$. The same pole is given by

$$g_2(t)g_3(t) : g_3(t)g_1(t) : g_1(t)g_2(t), \tag{19}$$

and Theorem 1 applies to form (19). With certain exceptions, the resulting conic (7) is a circumconic; specifically, if $uq_1 + rq_2 + qq_3 \neq 0$, then $g_1(t)$ has a root for which (19) is vertex A, and similarly for vertices B and C.

Example 6. For $P = u : v : w$, let $\Gamma(P)$ be the circumconic $u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$. Assume that at least one point of $\Gamma(P)$ lies inside $\triangle ABC$; in other words, assume that $\Gamma(P)$ is not an ellipse. Let $\widehat{\Gamma}(P)$ be the conic[6]

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0. \tag{20}$$

For each $\alpha : \beta : \gamma$ on the line $u\alpha + v\beta + w\gamma = 0$ and inside or on a side of $\triangle ABC$, let $P = p : q : r$, with $p \geq 0$, $q \geq 0$, $r \geq 0$, satisfy

$$\alpha = p^2, \quad \beta = q^2, \quad \gamma = r^2,$$

and define

$$\sqrt{P} := \sqrt{p} : \sqrt{q} : \sqrt{r} \tag{21}$$

and

$$P_A := -\sqrt{p} : \sqrt{q} : \sqrt{r}, \quad P_B := \sqrt{p} : -\sqrt{q} : \sqrt{r}, \quad P_C := \sqrt{p} : \sqrt{q} : -\sqrt{r} \tag{22}$$

Each point in (21) and (22) satisfies (20), and conversely, each point satisfying (20) is of one of the forms in (21) and (22). Therefore, the conic (20) consists

[6]Let $\Phi = vwa^2 + wub^2 + uvc^2$. Conic (20) is an ellipse, hyperbola, or parabola according as $\Phi > 0$, $\Phi < 0$, or $\Phi = 0$. Yff [6, pp.131-132], discusses a class of conics of the form (20) in connection with self-isogonal cubics and orthocentric systems.

of all points as in (21) and (22). Constructions[7] for $\sqrt{P}$ are known, and points P_A, P_B, P_C are constructible as harmonic conjugates involving $\sqrt{P}$ and vertices A, B, C; e.g., P_A is the harmonic conjugate of P with respect to A and the point $BC \cap AP$. Now suppose that L is a line, given by $\ell\alpha + m\beta + n\gamma = 0$. For X as in (18) traversing L, we have $g_1(t) = u(p_1 + q_1 t)$, leading to nine amenable coefficients in (4)-(6) and on to amenable cofactors, as indicated by

$$D_1 = up_1^2 r_1, \quad E_1 = -up_1 q_1 r_1, \quad F_1 = uq_1^2 r_1,$$

where $r_1 = p_2 q_3 - p_3 q_2$. The nine cofactors and (7) yield this conclusion: the Γ-based pole of X traverses the circumconic

$$\frac{\ell}{u\alpha} + \frac{m}{v\beta} + \frac{n}{w\gamma} = 0. \tag{23}$$

For example, taking line $u\alpha + v\beta + w\gamma = 0$ to be the trilinear polar of X_{100} and L that of X_{101}, the conic (23) is the Steiner circumellipse. In this case, the conic (20) is the hyperbola passing through X_i for $i = 1, 43, 165, 170, 365$, and 846. Another notable choice of (20) is given by $P = X_{798}$, which has first trilinear $(\cos^2 B - \cos^2 C)\sin^2 A$. Points on this hyperbola include X_i for $i = 1, 2, 20, 63, 147, 194, 478, 488, 616, 617, 627$, and 628.

Of course, for each $X = x : y : z$ on a conic $\widehat{\Gamma}(P)$, the points

$$-x : y : z, \quad x : -y : z, \quad x : y : -z$$

are also on $\widehat{\Gamma}(P)$, and if X also lies inside $\triangle ABC$, then X_1/X^2 lies on $\Gamma(P)$.

Example 7. Let Γ be the circumcircle, given by $a/\alpha + b/\beta + c/\gamma = 0$, and let L be the Brocard axis, which is the line passing through the points $X_6 = a : b : c$ and $X_3 = \cos A : \cos B : \cos C$. Using notation in Theorem 1, we find

$$d_1 = bc, \quad e_1 = 2a(b^2 + c^2), \quad f_1 = 4a^2 bc$$

and

$$D_1 = 8ab^2c^2(c^2 - b^2), \quad E_1 = 4a^2bc(b^2 - c^2), \quad F_1 = 2a^3(c^2 - b^2),$$

leading to this conclusion: the circumcircle-based pole of X traversing the Brocard axis traverses the circumhyperbola

$$\frac{a(b^2 - c^2)}{\alpha} + \frac{b(c^2 - a^2)}{\beta} + \frac{c(a^2 - b^2)}{\gamma} = 0,$$

namely, the isogonal transform of the trilinear polar of the Steiner point.

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[7]The trilinear square root is constructed in [4]. An especially attractive construction of barycentric square root in [7] yields a second construction of trilinear square root. We describe the latter here. Suppose $P = p : q : r$ in trilinears; then in barycentric, $P = ap : bq : cr$, so that the barycentric square root of P is $\sqrt{ap} : \sqrt{bq} : \sqrt{cr}$. Barycentric multiplication (as in [7]) by $\sqrt{a} : \sqrt{b} : \sqrt{c}$ gives $a\sqrt{p} : b\sqrt{q} : c\sqrt{r}$, these being barycentrics for the trilinear square root of P, which in trilinears is $\sqrt{p} : \sqrt{q} : \sqrt{r}$.

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