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Cubics Associated with Triangles of Equal Areas

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Abstract. The locus of a point X for which the cevian triangle of X and that of its isogonal conjugate have equal areas is a cubic that passes through the 1st and 2nd Brocard points. Generalizing from isogonal conjugate to P-isoconjugate yields a cubic $Z(U,P)$ passing through U; if X is on $Z(U,P)$ then the P-isoconjugate of X is on $Z(U,P)$ and this point is collinear with X and U. A generalized equal areas cubic $\Gamma(P)$ is presented as a special case of $Z(U,P)$. If $\sigma = \text{area}(\triangle ABC)$, then the locus of X whose cevian triangle has prescribed oriented area $K\sigma$ is a cubic $\Lambda(P)$, and P is determined if K has a certain form. Various points are proved to lie on $\Lambda(P)$.

1. Introduction

For any point $X = \alpha : \beta : \gamma$ (homogeneous trilinear coordinates) not a vertex of $\triangle ABC$, let

$$T = \begin{pmatrix} 0 & \beta & \gamma \\ \alpha & 0 & \gamma \\ \alpha & \beta & 0 \end{pmatrix} \qquad \text{and} \qquad T' = \begin{pmatrix} 0 & \gamma & \beta \\ \gamma & 0 & \alpha \\ \beta & \alpha & 0 \end{pmatrix},$$

so that T is the cevian triangle of X, and T' is the cevian triangle of the isogonal conjugate of X. Let σ be the area of $\triangle ABC$, and assume that X does not lie on a sideline $\triangle ABC$. Then oriented areas are given (e.g. [3, p.35]) in terms of the sidelengths a, b, c by

$$\text{area}(T) = \frac{abc}{8\sigma^2}\begin{vmatrix} 0 & k_1\beta & k_1\gamma \\ k_2\alpha & 0 & k_2\gamma \\ k_3\alpha & k_3\beta & 0 \end{vmatrix}, \qquad \text{area}(T') = \frac{abc}{8\sigma^2}\begin{vmatrix} 0 & l_1\gamma & l_1\beta \\ l_2\gamma & 0 & l_2\alpha \\ l_3\beta & l_3\alpha & 0 \end{vmatrix},$$

where k_i and l_i are normalizers. [1] Thus,

$$\text{area}(T) = \frac{k_1k_2k_3\alpha\beta\gamma abc}{4\sigma^2} \qquad \text{and} \qquad \text{area}(T') = \frac{l_1l_2l_3\alpha\beta\gamma abc}{8\sigma^2},$$

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[1] If $P = \alpha : \beta : \gamma$ is not on the line $\mathcal{L}^\infty$ at infinity, then the normalizer h makes $h\alpha, h\beta, h\gamma$ the directed distances from P to sidelines BC, CA, AB, respectively, and $h = 2\sigma/(a\alpha + b\beta + c\gamma)$. If P is on $\mathcal{L}^\infty$ and $\alpha\beta\gamma \neq 0$, then the normalizer is $h := 1/\alpha + 1/\beta + 1/\gamma$; if P is on $\mathcal{L}^\infty$ and $\alpha\beta\gamma = 0$, then $h := 1$.

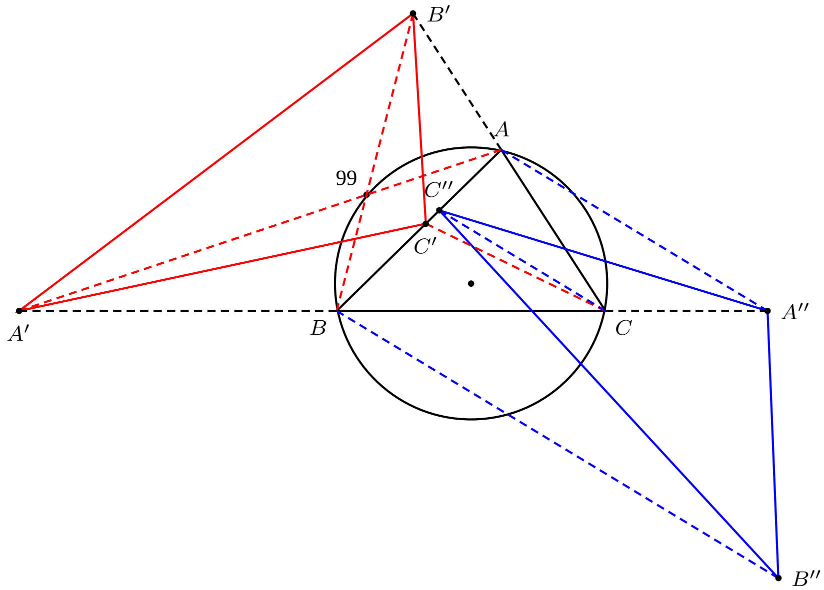

Figure 1. Triangles $A'B'C'$ and $A''B''C''$ have equal areas

so that area$(T) =$ area(T') if and only if $k_1k_2k_3 = l_1l_2l_3$. Substituting yields

$$\frac{1}{b\beta+c\gamma}\cdot\frac{1}{c\gamma+a\alpha}\cdot\frac{1}{a\alpha+b\beta}=\frac{1}{b\gamma+c\beta}\cdot\frac{1}{c\alpha+a\gamma}\cdot\frac{1}{a\beta+b\alpha},$$

which simplifies to

$$a(b^2-c^2)\alpha(\beta^2-\gamma^2)+b(c^2-a^2)\beta(\gamma^2-\alpha^2)+c(a^2-b^2)\gamma(\alpha^2-\beta^2)=0. \quad (1)$$

In the parlance of [4, p.240], equation (1) represents the self-isogonal cubic $Z(X_{512})$, and, in the terminology of [1, 2], the auto-isogonal cubic having pivot X_{512}. [2] It is easy to verify that the following 24 points lie on this cubic:[3]

vertices A, B, C,
incenter X_1 and excenters,
Steiner point X_{99} and its isogonal conjugate X_{512} (see Figure 1),
vertices of the cevian triangle of X_{512},
1st and 2nd Brocard points Ω_1 and Ω_2,
$X_{512}©X_1$ and $X_{512}©X_{99}$, where © denotes Ceva conjugate,
$(X_{512}©X_1)^{-1}$ and $(X_{512}©X_{99})^{-1}$, where $()^{-1}$ denotes isogonal conjugate,
vertices of triangle T_1 below,
vertices of triangle T_2 below.

[2] X_i is the ith triangle center as indexed in [5].

[3] This "equal-areas cubic" was the subject of a presentation by the author at the CRCC geometry meeting hosted by Douglas Hofstadter at Indiana University, March 23-25, 1999.

The vertices of the bicentric[4] triangle T_1 are

$$-ab : a^2 : bc, \qquad ca : -bc : b^2, \qquad c^2 : ab : -ca, \tag{2}$$

and those of T_2 are

$$-ac : bc : a^2, \qquad b^2 : -ba : ca, \qquad ab : c^2 : -cb. \tag{3}$$

Regarding (2), $-ab : a^2 : bc$ is the point other than A and Ω_1 in which line $A\Omega_1$ meets $Z(X_{512})$. Similarly, lines $A\Omega_1$ and $C\Omega_1$ meet $Z(X_{512})$ in the other two points in (2). Likewise, the points in (3) lie on lines $A\Omega_2$, $B\Omega_2, C\Omega_2$. The points in (3) are isogonal conjugates of those in (2).

Vertex $A' := -ab : a^2 : bc$ is the intersection of the C-side of the anticomplementary triangle and the B-exsymmedian, these being the lines $a\alpha + b\beta = 0$ and $c\alpha + a\gamma = 0$. The other five vertices are similarly constructed.

Other descriptions of $Z(X_{512})$ are easy to check: (i) the locus of a point Q collinear with its isogonal conjugate and X_{512}, and (ii) the locus of Q for which the line joining Q and its isogonal conjugate is parallel to the line $\Omega_1\Omega_2$.

2. Isoconjugates and reciprocal conjugates

In the literature, isoconjugates are defined in terms of trilinears and reciprocal conjugates are defined in terms of barycentrics. We shall, in this section, use the notations $(x : y : z)_t$ and $(x : y : z)_b$ to indicate trilinears and barycentrics, respectively. [5]

Definition 1. [6] Suppose $P = (p : q : r)_t$ and $X = (x : y : z)_t$ are points, neither on a sideline of $\triangle ABC$. The P-isoconjugate of X is the point

$$(P \cdot X)_t^{-1} = (qryz : rpzx : pqxy)_t.$$

On the left side, the subscript t signifies trilinear multiplication and division.

Definition 2. [3] Suppose $P = (p : q : r)_b$ and $X = (x : y : z)_b$ are points not on a sideline of $\triangle ABC$. The P-reciprocal conjugate of X is the point

$$(P/X)_b = (pyz : qzx : rxy)_b.$$

In keeping with the meanings of "iso-" and "reciprocal",

$$\begin{aligned} X\text{-isoconjugate of } P &= P\text{-isoconjugate of } X, \\ X\text{-reciprocal conjugate of } P &= \frac{G}{P\text{-reciprocal conjugate of } X}, \end{aligned}$$

where G, the centroid, is the identity corresponding to barycentric division.

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[4]Definitions of bicentric triangle, bicentric pair of points, and triangle center are given in [5, Glossary]. If $f(a, b, c) : g(a, b, c) : h(a, b, c)$ is the A-vertex of a bicentric triangle, then the B-vertex is $h(b, c, a) : f(b, c, a) : g(b, c, a)$ and the C-vertex is $g(c, a, b) : h(c, a, b) : f(c, a, b)$.

[5]A point X with trilinears $\alpha : \beta : \gamma$ has barycentrics $a\alpha : b\beta : c\gamma$. For points not on $\mathcal{L}^\infty$, trilinears are proportional to the directed distances between X and the sidelines BC, CA, AB, respectively, whereas barycentrics are proportional to the oriented areas of triangles XBC, XCA, XAB, respectively.

3. The cubic $Z(U,P)$

In this section, all coordinates are trilinears; for example, $(\alpha : \beta : \gamma)_t$ appears as $\alpha : \beta : \gamma$. Suppose $U = u : v : w$ and $P = p : q : r$ are points, neither on a sideline of $\triangle ABC$. We generalize the cubic $Z(U)$ defined in [4, p.240] to a cubic $Z(U,P)$, defined as the locus of a point $X = \alpha : \beta : \gamma$ for which the points U, X, and the P-isoconjugate of X are collinear. This requirement is equivalent to

$$\begin{vmatrix} u & v & w \\ \alpha & \beta & \gamma \\ qr\beta\gamma & rp\gamma\alpha & pq\alpha\beta \end{vmatrix} = 0, \tag{4}$$

hence to

$$up\alpha(q\beta^2 - r\gamma^2) + vq\beta(r\gamma^2 - p\alpha^2) + wr\gamma(q\alpha^2 - r\beta^2) = 0.$$

Equation (4) implies these properties:

(i) $Z(U,P)$ is self P-isoconjugate;
(ii) $U \in Z(U,P)$;
(iii) if $X \in Z(U,P)$, then X, U, and $(P \cdot X)_t^{-1}$ are collinear.

The following ten points lie on $Z(U,P)$:

the vertices A, B, C;
the vertices of the cevian triangle of U, namely,

$$0 : v : w, \quad u : 0 : w, \qquad u : v : 0; \tag{5}$$

and the points invariant under P-isoconjugation:

$$\frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \tag{6}$$

$$\frac{-1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{p}} : \frac{-1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{-1}{\sqrt{r}}. \tag{7}$$

As an illustration of (i), the cubics $Z(U,X_1)$ and $Z(U,X_{31})$ are self-isogonal conjugate and self-isotomic conjugate, respectively. Named cubics of the type $Z(U,X_1)$ include the Thomson ($U = X_2$), Darboux ($U = X_{20}$), Neuberg ($U = X_{30}$), Ortho ($U = X_4$), and Feuerbach ($U = X_5$). The Lucas cubic is $Z(X_{69}, X_{31})$, and the Spieker, $Z(X_8, X_{58})$. Table 1 offers a few less familiar cubics.

It is easy to check that the points

$$U \copyright X_1 = -u + v + w : u - v + w : u + v - w,$$
$$U \copyright U^{-1} = \frac{1}{u}(-\frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2}) : \frac{1}{v}(\frac{1}{u^2} - \frac{1}{v^2} + \frac{1}{w^2}) : \frac{1}{w}(\frac{1}{u^2} + \frac{1}{v^2} - \frac{1}{w^2})$$

lie on $Z(U)$. Since their isogonal conjugates also lie on $Z(U)$, we have four more points on $Z(U,P)$ in the special case that $P = X_1$.

U	P	Centers on cubic $Z(U,P)$
X_{385}	X_1	$X_1, X_2, X_6, X_{32}, X_{76}, X_{98}, X_{385}, X_{511}, X_{694}$
X_{395}	X_1	$X_1, X_2, X_6, X_{14}, X_{16}, X_{18}, X_{62}, X_{395}$
X_{396}	X_1	$X_1, X_2, X_6, X_{13}, X_{15}, X_{17}, X_{61}, X_{396}$
X_{476}	X_1	$X_1, X_{30}, X_{74}, X_{110}, X_{476}, X_{523}, X_{526}$
X_{171}	X_2	$X_2, X_{31}, X_{42}, X_{43}, X_{55}, X_{57}, X_{81}, X_{171}, X_{365}, X_{846}, X_{893}$
X_{894}	X_6	$X_6, X_7, X_9, X_{37}, X_{75}, X_{86}, X_{87}, X_{192}, X_{256}, X_{366}, X_{894}, X_{1045}$
X_{309}	X_{31}	$X_2, X_{40}, X_{77}, X_{189}, X_{280}, X_{309}, X_{318}, X_{329}, X_{347}, X_{962}$
X_{226}	X_{55}	$X_2, X_{57}, X_{81}, X_{174}, X_{226}, X_{554}, X_{559}, X_{1029}, X_{1081}, X_{1082}$
X_{291}	X_{239}	$X_1, X_6, X_{42}, X_{57}, X_{239}, X_{291}, X_{292}, X_{672}, X_{894}$
X_{292}	X_{238}	$X_1, X_2, X_{37}, X_{87}, X_{171}, X_{238}, X_{241}, X_{291}, X_{292}$

Table 1

4. Trilinear generalization: $\Gamma(P)$

Next we seek the locus of a point $X = \alpha : \beta : \gamma$ (trilinears) for which the cevian triangle T and the cevian triangle

$$\widehat{T} = \begin{pmatrix} 0 & r\gamma & q\beta \\ r\gamma & 0 & p\alpha \\ q\beta & p\alpha & 0 \end{pmatrix}$$

of the P-isoconjugate of X have equal areas. For this, the method leading to (1) yields a cubic denoted by $\Gamma(P)$:

$$ap(rb^2-qc^2)\alpha(q\beta^2-r\gamma^2)+bq(pc^2-ra^2)\beta(r\gamma^2-p\alpha^2)+cr(qa^2-pb^2)\gamma(p\alpha^2-q\beta^2)=0, \tag{8}$$

except for $P = X_{31} = a^2 : b^2 : c^2$; that is, except when P-isoconjugation is isotomic conjugation, for which the two triangles have equal areas for all X. The cubic (8) is $Z(U,P)$ for

$$U = U(P) = a(rb^2 - qc^2) : b(pc^2 - ra^2) : c(qa^2 - pb^2),$$

a point on $\mathcal{L}^\infty$. As in Section 3, the vertices A, B, C and the points (5)-(7) lie on $\Gamma(P)$.

Let U^* denote the P-isoconjugate of U. This is the trilinear pole of the line XX_2, where $X = \frac{a}{p} : \frac{b}{q} : \frac{c}{r}$, the P-isoconjugate of X_2. Van Lamoen has noted that since U lies on the trilinear polar, L, of the P-isoconjugate of the centroid (i.e., L has equation $\frac{p\alpha}{a} + \frac{q\beta}{b} + \frac{r\gamma}{c} = 0$), and U also lies on $\mathcal{L}^\infty$, we have U^* lying on the Steiner circumellipse and on the conic

$$\frac{pa}{\alpha} + \frac{qb}{\beta} + \frac{rc}{\gamma} = 0, \tag{9}$$

this being the P-isoconjugate of $\mathcal{L}^\infty$.

Theorem 1. *Suppose P_1 and P_2 are distinct points, collinear with but not equal to X_{31}. Then $U(P_2) = U(P_1)$.*

Proof. Write $P_1 = p_1 : q_1 : r_1$ and $P_2 = p_2 : q_2 : r_2$. Then for some $s = s(a,b,c) \neq 0$,

$$a^2 = sp_1 + p_2, \qquad b^2 = sq_1 + q_2, \qquad c^2 = sr_1 + r_2,$$

so that for $f(a,b,c) := a[(c^2 - sr_1)b^2 - (b^2 - sq_1)c^2]$, we have

$$\begin{aligned} U(P_2) &= f(a,b,c) : f(b,c,a) : f(c,a,b) \\ &= a(sc^2q_1 - sb^2r_1) : b(sa^2r_1 - sc^2p_1) : c(sb^2p_1 - sa^2q_1) \\ &= U(P_1). \end{aligned}$$

□

Example 1. For each point P on the line $X_1X_{31,}$ the pivot $U(P)$ is the isogonal conjugate (X_{512}) of the Steiner point (X_{99}). Such points P include the Schiffler point (X_{21}), the isogonal conjugate (X_{58}) of the Spieker center, and the isogonal conjugate (X_{63}) of the Clawson point.

The cubic $\Gamma(P)$ meets $\mathcal{L}^\infty$ in three points. Aside from U, the other two are where $\mathcal{L}^\infty$ meets the conic (9). If (9) is an ellipse, then the two points are nonreal. In case P is the incenter, so that the cubic is the equal areas cubic, the two points are given in [6, p.116] by the ratios [6]

$$e^{\pm iB} : e^{\mp iA} : -1.$$

Theorem 2. *The generalized Brocard points defined by*

$$\frac{qc}{b} : \frac{ra}{c} : \frac{pb}{a} \qquad \text{and} \qquad \frac{rb}{c} : \frac{pc}{a} : \frac{qa}{b} \tag{10}$$

lie on $\Gamma(P)$.

Proof. Writing ordered triples for the two points, we have

$$\begin{aligned} & (a(rb^2 - qc^2),\ b(pc^2 - ra^2),\ c(qa^2 - pb^2)) \\ = \quad & abc(\frac{qc}{b}, \frac{ra}{c}, \frac{pb}{a}) + abc(\frac{rb}{c}, \frac{pc}{a}, \frac{qa}{b}), \end{aligned}$$

showing U as a linear combination of the points in (10). Since those two are isogonal conjugates collinear with U, they lie on $\Gamma(P)$. □

If P is a triangle center, then the generalized Brocard points (10) comprise a bicentric pair of points. In §8, we offer geometric constructions for such points.

[6]The pair is also given by $-1 : e^{\pm iC} : e^{\mp iB}$ and by $e^{\mp iC} : -1 : e^{\pm iA}$. Multiplying the three together and then by -1 gives cubes in "central form" with first coordinates

$$\cos(B - C) \pm i \sin(B - C).$$

The other coordinates are now given from the first by cyclic permutations.

5. Barycentric generalization: $\hat{\Gamma}(P)$

Here, we seek the locus of a point $X = \alpha : \beta : \gamma$ (barycentrics) for which the cevian triangle of the P-reciprocal conjugate of X and that of X have equal areas. The method presented in §1 yields a cubic that we denote by $\hat{\Gamma}(P)$:

$$p(q-r)\alpha(r\beta^2 - q\gamma^2) + q(r-p)\beta(p\gamma^2 - r\alpha^2) + r(p-q)\gamma(q\alpha^2 - p\beta^2) = 0, \quad (11)$$

In particular, the equal areas cubic (1) is given by (11) using

$$(p : q : r)_b = (a^2 : b^2 : c^2)_b.$$

In contrast to (11), if equation (1) is written as $s(a, b, c, \alpha, \beta, \gamma) = 0$, then

$$s(\alpha, \beta, \gamma, a, b, c) = s(a, b, c, \alpha, \beta, \gamma),$$

a symmetry stemming from the use of trilinear coordinates and isogonal conjugation.

6. A sextic

For comparison with the cubic $\Gamma(P)$ of §4, it is natural to ask about the locus of a point X for which the anticevian triangle of X and that of its isogonal conjugate have equal areas. The result is easily found to be the self-isogonal sextic

$$\begin{aligned} & \alpha\beta\gamma(-a\alpha + b\beta + c\gamma)(a\alpha - b\beta + c\gamma)(a\alpha + b\beta - c\gamma) \\ = \; & (-a\beta\gamma + b\gamma\alpha + c\alpha\beta)(a\beta\gamma - b\gamma\alpha + c\alpha\beta)(a\beta\gamma + b\gamma\alpha - c\alpha\beta), \end{aligned}$$

on which lie A, B, C, the incenter, excenters, and the two Brocard points. Remarkably, the vertices A, B, C are triple points of this sextic.

7. Prescribed area cubic: $\Lambda(P)$

Suppose $P = p : q : r$ (trilinears) is a point, and let $K\sigma$ be the oriented area of the cevian triangle of P. The method used in §1 shows that if $X = \alpha : \beta : \gamma$, then the cevian triangle of X has area $K\sigma$ if

$$k_1 k_2 k_3 abc\alpha\beta\gamma = 8K\sigma^3, \quad (12)$$

where $k_1 = \frac{2\sigma}{b\beta + c\gamma}$ and k_2 and k_3 are obtained cyclically. Substituting into (12) and simplifying gives

$$K = 2 \cdot \frac{pa}{bq + cr} \cdot \frac{qb}{cr + ap} \cdot \frac{rc}{ap + bq}. \quad (13)$$

The locus of X for which (13) holds is therefore given by the equation

$$(bq + cr)(cr + ap)(ap + bq)\alpha\beta\gamma - pqr(b\beta + c\gamma)(c\gamma + a\alpha)(a\alpha + b\beta) = 0. \quad (14)$$

We call this curve the *prescribed area cubic for* P (or for K) and denote it by $\Lambda(P)$. One salient feature of $\Lambda(P)$, easily checked by substituting

$$\frac{1}{a^2\alpha}, \; \frac{1}{b^2\beta}, \; \frac{1}{c^2\gamma}$$

for α, β, γ, respectively, into the left side of (14), is that $\Lambda(P)$ is self-isotomic. That is, if X lies on $\Lambda(P)$ but not on a sideline of $\triangle ABC$, then so does its isotomic conjugate, which we denote by $\widetilde{X}$. (Of course, we already know that $\Lambda(P)$ is self-isotomic, by the note just after (8)).

If $(bq-cr)(cr-ap)(ap-bq) \neq 0$, then the line $P\widetilde{P}$ meets $\Lambda(P)$ in three points, namely P, $\widetilde{P}$, and the point

$$P' := \frac{a^2p^2 - bcqr}{a^2p(bq-cr)} \; : \; \frac{b^2q^2 - carp}{b^2q(cr-ap)} \; : \; \frac{c^2r^2 - abpq}{c^2r(ap-bq)}.$$

If P is a triangle center on $\Lambda(P)$, then $\widetilde{P}$, P', and $\widetilde{P'}$ are triangle centers on $\Lambda(P)$. Since $\widetilde{P'}$ is not collinear with the others, three triangle centers on $\Lambda(P)$ can be found as points of intersection of $\Lambda(P)$ with the lines joining $\widetilde{P'}$ to P, $\widetilde{P}$, and P'. Then more central lines are defined, bearing triangle centers that lie on $\Lambda(P)$, and so on. Some duplication of centers thus defined inductively can be expected, but one wonders if, for many choices of P, this scheme accounts for infinitely many centers lying on $\Lambda(P)$.

It is easy to check that $\Lambda(P)$ meets the line at infinity in the following points:

$$A' := 0 : c : -b, \qquad B' := -c : 0 : a, \qquad C' := b : -a : 0.$$

Three more points are found by intersecting lines PA', PB', PC' with $\Lambda(P)$:

$$A'' := bcp : c^2r : b^2q, \quad B'' := c^2r : caq : a^2p, \quad C'' := b^2q : a^2p : abr.$$

A construction for A'' is given by the equation $A'' = PA' \cap \widetilde{P}A$.

Line AP meets $\Lambda(P)$ in the collinear points A, P, and, as is easily checked, the point

$$\frac{bcqr}{pa^2} : q : r.$$

Writing this and its cyclical cousins integrally, we have these points on $\Lambda(P)$:

$$bcqr : a^2pq : a^2rp, \qquad b^2pq : carp : b^2qr, \qquad c^2rp : c^2qr : abpq.$$

We have seen for given P how to form K. It is of interest to reverse these. Suppose a prescribed area is specified as $K\sigma$, where K has the form

$$k(a,b,c)k(b,c,a)k(c,a,b)$$

in which $k(a,b,c)$ is homogeneous of degree zero in a, b, c.[7] We abbreviate the factors as k_a, k_b, k_c and seek a point $P = p : q : r$ satisfying

$$K = k_ak_bk_c = \frac{2abcpqr}{(bq+cr)(cr+ap)(ap+bq)}.$$

Solving the system obtained cyclically from

$$k_a = \frac{\sqrt[3]{2}ap}{bq+cr} \tag{15}$$

yields

$$p : q : r = \frac{k_a}{a(\sqrt[3]{2}+k_a)} : \frac{k_b}{b(\sqrt[3]{2}+k_b)} : \frac{k_c}{c(\sqrt[3]{2}+k_c)}$$

[7] That is, $k(ta,tb,tc) = k(a,b,c)$, where t is an indeterminate.

except for $k_a = -\sqrt[3]{2}$, which results from (15) with $P = X_{512}$. The following table offers a variety of examples:

P	$k_a/\sqrt[3]{2}$
X_1	$\frac{a}{b+c}$
X_2	1
X_3	$\frac{\sin 2A}{\sin 2B+\sin 2C}$
X_4	$\frac{\tan A}{\tan B+\tan C}$
X_{10}	$\frac{b+c}{a}$
X_{57}	$-\frac{a}{b+c}$
X_{870}	$\frac{bc}{b^2+c^2}$
X_{873}	$\frac{2bc}{b^2+c^2}$

Table 2

Next, suppose $U = u : v : w$ is a point, not on a sideline of $\triangle ABC$, and let

$$P = \frac{vc}{b} : \frac{wa}{c} : \frac{ub}{a}.$$

Write out K as in (13), and use not (15), but instead, put

$$k_a = \frac{\sqrt[3]{2}a^2u}{b^2w + c^2v},$$

corresponding to the point $U \cdot X_6 = ua : vb : wc$, in the sense that the cevian triangle of $U \cdot X_6$ and that of P have equal areas. Likewise, the cevian triangle of the point

$$P' = \frac{wb}{c} : \frac{uc}{a} : \frac{va}{b},$$

has the same area, $K\sigma$. The points P and P' are essentially those of Theorem 2.

Three special cases among the cubics $\Lambda(P)$ deserve further comment. First, for $K = 2$, corresponding to $P = X_{512}$, equation (14) takes the form

$$(a\alpha + b\beta + c\gamma)(bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta) = 0. \tag{16}$$

Since $\mathcal{L}^\infty$ is given by the equation $a\alpha + b\beta + c\gamma = 0$ and the Steiner circumellipse is given by

$$bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0,$$

the points satisfying (16) occupy the line and the ellipse together. J.H. Weaver [8] discusses the cubic.

Second, when $K = \frac{1}{4}$, the cubic $\Lambda(P)$ is merely a single point, the centroid. Finally, we note that $\Lambda(X_6)$ passes through these points:

$$a : b : c, \quad a : c : b, \quad b : c : a, \quad b : a : c, \quad c : a : b, \quad c : b : a. \tag{17}$$

8. Constructions

In the preceding sections, certain algebraically defined points, as in (17), have appeared. In this section, we offer Euclidean constructions for such points.For given $U = u : v : w$ and $X = x : y : z$ and let us begin with the trilinear product, quotient, and square root, denoted respectively by $U \cdot X$, U/X, and $\sqrt{X}$.

Constructions for closely related barycentric product, quotient, and square root are given in [9], and these constructions are easily adapted to give the trilinear results.

We turn now to a construction from X of the point $x : z : y$. In preparation, decree as *positive* the side of line AB that contains C, and also the side of line CA that contains B. The opposite sides will be called *negative*.

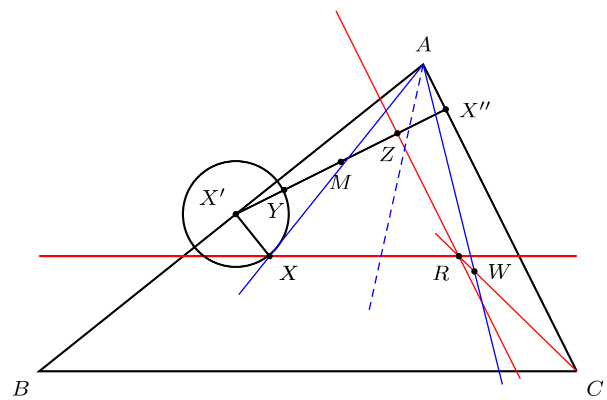

Figure 2. Construction of $W = x : z : y$ from $X = x : y : z$

Let X' be the foot of the perpendicular from X on line AB, and let X'' be the foot of the perpendicular from X' on line CA. Let M be the midpoint of segment $X'X''$, and let $\mathcal{O}$ be the circle centered at X' and passing through X. Line $X'X''$ meets circle $\mathcal{O}$ in two points; let Y be the one closer to M, as in Figure 2, and let Z' be the reflection of Y in M. If X is on the positive side of AB and Z' is on the positive side of CA, or if X is on the negative side of AB and Z' is on the negative side of CA, then let $Z = Z'$; otherwise let Z be the reflection of Z' in line CA.

Now line L through Z parallel to line CA has directed distance kz from line CA, where kx is the directed distance from line BC of the line L' through X parallel to BC. Let $R = L \cap L'$. Line CR has equation $z\alpha = x\beta$. Let L'' be the reflection of line AX about the internal angle bisector of $\angle CAB$. This line has equation $y\beta = z\gamma$. Geometrically and algebraically, it is clear that $x : z : y = CR \cap L''$, labeled W in Figure 2.

One may similarly construct the point $y : z : x$ as the intersection of lines $x\beta = z\gamma$ and $z\alpha = y\beta$. Then any one of the six points

$$x : y : z, \quad x : z : y, \quad y : z : x, \quad y : x : z, \quad z : x : y, \quad z : y : x,$$

can serve as a starting point for constructing the other five. (A previous appearance of these six points is [4, p.243], where an equation for the Yff conic, passing through the six points, is given.)

The methods of this section apply, in particular, to the constructing of the generalized Brocard points (10); e.g., for given $P = p : q : r$, construct $P' := q : r : p$, and then construct $P' \cdot \Omega_1$.

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