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Some Properties of the Lemoine Point

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Abstract. The Lemoine point, K, of $\triangle ABC$ has special properties involving centroids of pedal triangles. These properties motivate a definition of Lemoine field, F, and a coordinatization of the plane of $\triangle ABC$ using perpendicular axes that pass through K. These principal axes are symmetrically related to two other lines: one passing through the isodynamic centers, and the other, the isogonic centers.

1. Introduction

Let $A'B'C'$ be the pedal triangle of an arbitrary point Z in the plane of a triangle ABC, and consider the vector field $\mathbf{F}$ defined by $\mathbf{F}(Z) = \mathbf{ZA}' + \mathbf{ZB}' + \mathbf{ZC}'$. It is well known that $\mathbf{F}(Z)$ is the zero vector if and only if Z is the Lemoine point, K, also called the symmedian point. We call $\mathbf{F}$ the *Lemoine field* of $\triangle ABC$ and K the *balance point* of $\mathbf{F}$.

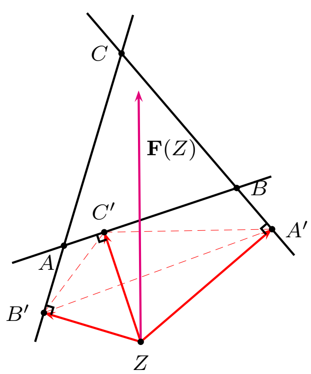

Figure 1

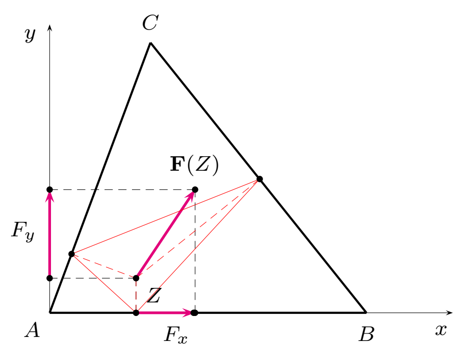

Figure 2

The Lemoine field may be regarded as a physical force field. Any point Z in this field then has a natural motion along a certain curve, or trajectory. See Figure 1. We shall determine parametric equations for these trajectories and find, as a result, special properties of the lines that bisect the angles between the line of the isogonic centers and the line of the isodynamic centers of $\triangle ABC$.

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Я благодарю дорогую Лену за оказанную мне моральную поддержку. Также я очень признателен профессору Кимберлингу за разрешение многочисленных проблем, касающихся англииского языка. The author dedicates his work to *Helen* and records his appreciation to the Communicating Editor for assistance in translation.

2. The Lemoine equation

In the standard cartesian coordinate system, place $\triangle ABC$ so that $A = (0,0)$, $B = (c,0)$, $C = (m,n)$, and write $Z = (x,y)$. For any line $Px + Qy + R = 0$, the vector H from Z to the projection of Z on the line has components

$$h_x = \frac{-P}{P^2+Q^2}(Px+Qy+R), \qquad h_y = \frac{-Q}{P^2+Q^2}(Px+Qy+R).$$

From these, one find the components of the three vectors whose sum defines $\mathbf{F}(Z)$:

vector	$x -$ component	$y -$ component
$\mathbf{ZA}'$	$\frac{-n(nx+y(c-m)-cn)}{n^2+(c-m)^2}$	$\frac{(m-c)(nx+y(c-m)-cn)}{n^2+(c-m)^2}$
$\mathbf{ZB}'$	$\frac{-n(nx-my)}{m^2+n^2}$	$\frac{m(nx-my)}{m^2+n^2}$
$\mathbf{ZC}'$	0	$-y$

The components of the Lemoine field $\mathbf{F}(Z) = \mathbf{ZA}' + \mathbf{ZB}' + \mathbf{ZC}'$ are given by

$$F_x = -(\alpha x + \beta y) + d_x, \qquad F_y = -(\beta x + \gamma y) + d_y,$$

where

$$\begin{aligned}
&\alpha = \tfrac{n^2}{m^2+n^2} + \tfrac{n^2}{n^2+(c-m)^2}, && \beta = \tfrac{-mn}{m^2+n^2} + \tfrac{n(c-m)}{n^2+(c-m)^2},\\
&\gamma = 1 + \tfrac{m^2}{m^2+n^2} + \tfrac{(c-m)^2}{n^2+(c-m)^2}; &&\\
&d_x = \tfrac{cn^2}{n^2+(c-m)^2}, && d_y = \tfrac{cn(c-m)}{n^2+(c-m)^2}.
\end{aligned}$$

See Figure 2. Assuming a unit mass at each point Z, Newton's Second Law now gives a system of differential equations:

$$x'' = -(\alpha x + \beta y) + d_x, \qquad y'' = -(\beta x + \gamma y) + d_y,$$

where the derivatives are with respect to time, t. We now translate the origin from $(0,0)$ to the balance point (d_x, d_y), which is the Lemoine point K, thereby obtaining the system

$$x'' = -(\alpha x + \beta y), \qquad y'' = -(\beta x + \gamma y),$$

which has the matrix form

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -M \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$. We shall refer to (1) as the *Lemoine equation.*

3. Eigenvalues of the matrix M

In order to solve equation (1), we first find eigenvalues λ_1 and λ_2 of M. These are the solutions of the equation $|M - \lambda I| = 0$, *i.e.*, $(\alpha - \lambda)(\gamma - \lambda) - \beta^2 = 0$, or

$$\lambda^2 - (\alpha + \gamma)\lambda + (\alpha\gamma - \beta^2) = 0.$$

Thus

$$\lambda_1+\lambda_2=\alpha+\gamma=1+\frac{m^2+n^2}{m^2+n^2}+\frac{n^2+(c-m)^2}{n^2+(c-m)^2}=3.$$

Writing a, b, c for the sidelengths $|BC|$, $|CA|$, $|AB|$ respectively, we find the determinant

$$|M|=\alpha\gamma-\beta^2=\frac{n^2}{a^2b^2}(a^2+b^2+c^2)>0.$$

The discriminant of the characteristic equation $\lambda^2-(\alpha+\gamma)\lambda+(\alpha\gamma-\beta^2)=0$ is given by

$$D=(\alpha+\gamma)^2-4(\alpha\gamma-\beta^2)=(\alpha-\gamma)^2+4\beta^2\geq 0. \tag{2}$$

Case 1: equal eigenvalues $\lambda_1=\lambda_2=\frac{3}{2}$. In this case, $D=0$ and (2) yields $\beta=0$ and $\alpha=\gamma$. To reduce notation, write $p=c-m$. Then since $\beta=0$, we have $\frac{m}{m^2+n^2}=\frac{p}{p^2+n^2}$, so that

$$(m-p)(mp-n^2)=0. \tag{3}$$

Also, since $\alpha=\gamma$, we find after mild simplifications

$$n^4-(m^2+p^2)n^2-3m^2p^2=0. \tag{4}$$

Equation (3) imples that $m=p$ or $mp=n^2$. If $m=p$, then equation (4) has solutions $n=\sqrt{3}m=\sqrt{3}p$. Consequently, $C=\left(\frac{1}{2}c,\ \frac{\sqrt{3}}{2}c\right)$, so that $\triangle ABC$ is equilateral. However, if $mp=n^2$, then equation (4) leads to $(m+p)^2=0$, so that $c=0$, a contradiction. Therefore from equation (3) we obtain this conclusion: *if the eigenvalues are equal, then $\triangle ABC$ is equilateral.*

Case 2: distinct eigenvalues $\lambda_{1,2}=\frac{3\pm\sqrt{D}}{2}$. Here $D>0$, and $\lambda_{1,2}>0$ according to (2). We choose to consider the implications when

$$\beta=0, \qquad \alpha\neq\gamma. \tag{5}$$

We omit an easy proof that these conditions correspond to $\triangle ABC$ being a right triangle or an isosceles triangle. In the former case, write $c^2=a^2+b^2$. Then the characteristic equation yields eigenvalues α and γ, and

$$\alpha=\frac{n^2}{b^2}+\frac{n^2}{a^2}=\frac{n^2(a^2+b^2)}{a^2b^2}=\frac{n^2c^2}{a^2b^2}=1,$$

since $ab=nc=$ twice the area of the right triangle. Since $\alpha+\gamma=3$, $\gamma=2$.

4. General solution of the Lemoine equation

According to a well known theorem of linear algebra, rotation of the coordinate system about K gives the system $x''=-\lambda_1 x$, $y''=-\lambda_2 y$. Let us call the axes of this coordinate system the *principal axes* of the Lemoine field.

Note that if $\triangle ABC$ is a right triangle or an isosceles triangle (cf. conditions (5)), then the angle of rotation is zero, and K is on an altitude of the triangle. In this case, one of the principal axes is that altitude, and the other is parallel to the corresponding side. Also if $\triangle ABC$ is a right triangle, then K is the midpoint of that altitude.

In the general case, the solution of the Lemoine equation is given by

$$x = c_1 \cos \omega_1 t + c_2 \sin \omega_2 t, \qquad y = c_3 \cos \omega_1 t + c_4 \sin \omega_2 t, \tag{6}$$

where $\omega_1 = \sqrt{\lambda_1}$, $\omega_2 = \sqrt{\lambda_2}$. Initial conditions $x(0) = x_0$, $y(0) = y_0$, $x'(0) = 0$, $y'(0) = 0$ reduce (6) to

$$x = x_0 \cos \omega_1 t, \qquad y = y_0 \cos \omega_2 t, \tag{7}$$

with $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1^2 + \omega_2^2 = 3$. Equations (7) show that each trajectory is bounded. If $\lambda_1 = \lambda_2$, then the trajectory is a line segment; otherwise, (7) represents a Lissajous curve or an almost-everywhere rectangle-filling curve, according as $\frac{\omega_1}{\omega_2}$ is rational or not.

5. Lemoine sequences and centroidal orbits

Returning to the Lemoine field, **F**, suppose Z_0 is an arbitrary point, and G_{Z_0} is the centroid of the pedal triangle of Z_0. Let Z_0' be the point to which **F** translates Z_0. It is well known that G_{Z_0} lies on the line Z_0Z_0' at a distance $\frac{1}{3}$ of that from Z_0 to Z_0'. With this in mind, define inductively the *Lemoine sequence* of Z_0 as the sequence $(Z_0, Z_1, Z_2, \ldots)$, where Z_n, for $n \geq 1$, is the centroid of the pedal triangle of Z_{n-1}. Writing the centroid of the pedal triangle of Z_0 as $Z_1 = (x_1, y_1)$, we obtain $3(x_1 - x_0) = -\lambda_1 x_0$ and

$$x_1 = \frac{1}{3}(3 - \lambda_1)x_0 = \frac{1}{3}\lambda_2 x_0; \qquad y_1 = \frac{1}{3}\lambda_1 y_0.$$

Accordingly, the Lemoine sequence is given with respect to the principal axes by

$$Z_n = \left(x_0 \left(\frac{\lambda_2}{3} \right)^n, \; y_0 \left(\frac{\lambda_1}{3} \right)^n \right). \tag{8}$$

Since $\frac{1}{3}\lambda_1$ and $\frac{1}{3}\lambda_2$ are between 0 and 1, the points Z_n approach $(0,0)$ as $n \to \infty$. That is, the Lemoine sequence of every point converges to the Lemoine point.

Representation (8) shows that Z_n lies on the curve $(x, y) = (x_0 u^t, y_0 v^t)$, where $u = \frac{1}{3}\lambda_2$ and $v = \frac{1}{3}\lambda_1$. We call this curve the *centroidal orbit* of Z_0. See Figure 3. Reversing the directions of axes if necessary, we may assume that $x_0 > 0$ and $y_0 > 0$, so that elimination of t gives

$$\frac{y}{y_0} = \left(\frac{x}{x_0} \right)^k, \qquad k = \frac{\ln v}{\ln u}. \tag{9}$$

Equation (9) expresses the centroidal orbit of $Z_0 = (x_0, y_0)$. Note that if $\omega_1 = \omega_2$, then $v = u$, and the orbit is a line. Now let X_Z and Y_Z be the points in which line ZG_Z meets the principal axes. By (8),

$$\frac{|ZG_Z|}{|G_Z X_Z|} = \frac{\lambda_2}{\lambda_1}, \qquad \frac{|ZG_Z|}{|G_Z Y_Z|} = \frac{\lambda_1}{\lambda_2}. \tag{10}$$

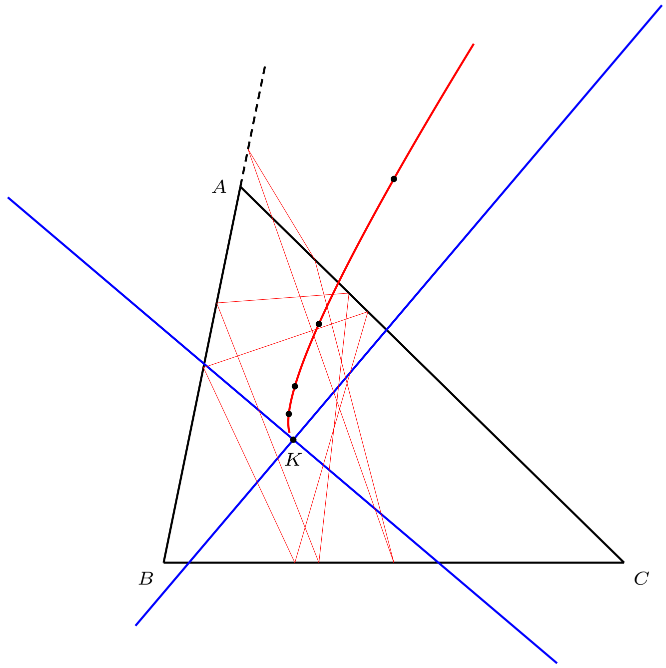

Figure 3

These equations imply that if $\triangle ABC$ is equilateral with center O, then the centroid G_Z is the midpoint of segment OG_Z.

As another consequence of (10), suppose $\triangle ABC$ is a right triangle; let H be the line parallel to the hypotenuse and passing through the midpoint of the altitude H' to the hypotenuse. Let X and Y be the points in which line ZG_Z meets H and H', respectively. Then $|ZG_Z| : |XG_Z| = |YG_Z| : |ZG_Z| = 2 : 1$.

6. The principal axes of the Lemoine field

Physically, the principal axes may be described as the locus of points in the plane of $\triangle ABC$ along which the "direction" of the Lemoine sequence remains constant. That is, if Z_0 lies on one of the principal axes, then all the points $Z_1, Z_2, \ldots$ lie on that axis also.

In this section, we turn to the geometry of the principal axes. Relative to the coordinate system adopted in §5, the principal axes have equations $x = 0$ and $y = 0$. Equation (8) therefore shows that if Z_0 lies on one of these two perpendicular lines, then Z_n lies on that line also, for all $n \geq 1$.

Let A_1, A_2 denote the isodynamic points, and F_1, F_2 the isogonic centers, of $\triangle ABC$. Call lines A_1A_2 and F_1F_2 the *isodynamic axis* and the *isogonic axis* respectively. [1]

Lemma 1. *Suppose Z and Z' are a pair of isogonal conjugate points. Let O and O' be the circumcircles of the pedal triangles of Z and Z'. Then $O = O'$, and the center of O is the midpoint between Z and Z'.*

[1]The points F_1, F_2, A_1, A_2 are indexed as X_{13}, X_{14}, X_{15}, X_{16} and discussed in [2].

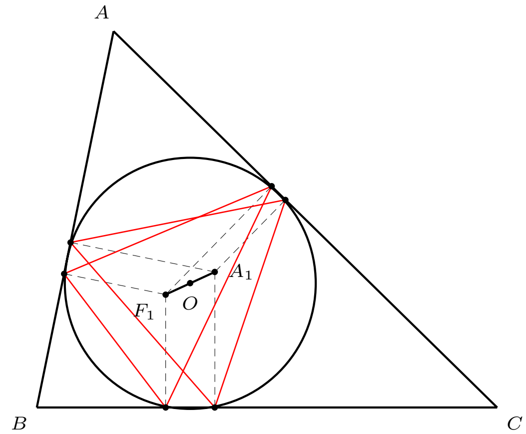

Figure 4

A proof is given in Johnson [1, pp.155–156]. See Figure 4.

Now suppose that $Z = A_1$. Then $Z' = F_1$, and, according to Lemma 1, the pedal triangles of Z and Z' have the same circumcircle, whose center O is the midpoint between A_1 and F_1. Since the pedal triangle of A_1 is equilateral, the point O is the centroid of the pedal triangle of A_1.

Next, suppose L is a line not identical to either of the principal axes. Let L' be the reflection of L about one of the principal axes. Then L' is also the reflection of L about the other principal axis. We call L and L' a *symmetric pair of lines*.

Lemma 2. *Suppose that G_P is the centroid of the pedal triangle of a point P, and that Q is the reflection of P in G_P. Then there exists a symmetric pair of lines, one passing through P and the other passing through Q.*

Proof. With respect to the principal axes, write $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. Then $G_P = (\frac{1}{3}\lambda_2 x_P,\ \frac{1}{3}\lambda_1 y_P)$, and $\frac{2}{3}\lambda_2 x_P = x_P + x_Q$, so that

$$x_Q = \left(\frac{2}{3}\lambda_2 - 1\right) x_P = \frac{1}{3}(2\lambda_2 - (\lambda_1 + \lambda_2))x_P = \frac{1}{3}(\lambda_2 - \lambda_1)x_P.$$

Likewise, $y_Q = \frac{1}{3} y_P(\lambda_1 - \lambda_2)$. It follows that $\frac{x_P}{y_P} = -\frac{x_Q}{y_Q}$. This equation shows that the line $y = \frac{y_P}{x_P} \cdot x$ passing through P and the line $y = \frac{y_Q}{x_Q} \cdot x$ passing through Q are symmetric about the principal axes $y = 0$ and $x = 0$. See Figure 5. $\square$

Theorem. *The principal axes of the Lemoine field are the bisectors of the angles formed at the intersection of the isodynamic and isogonic axes in the Lemoine point.*

Proof. In Lemma 2, take $P = A_1$ and $Q = F_1$. The symmetric pair of lines are then the isodynamic and isogonic axes. Their symmetry about the principal axes is equivalent to the statement that these axes are the asserted bisectors. $\square$

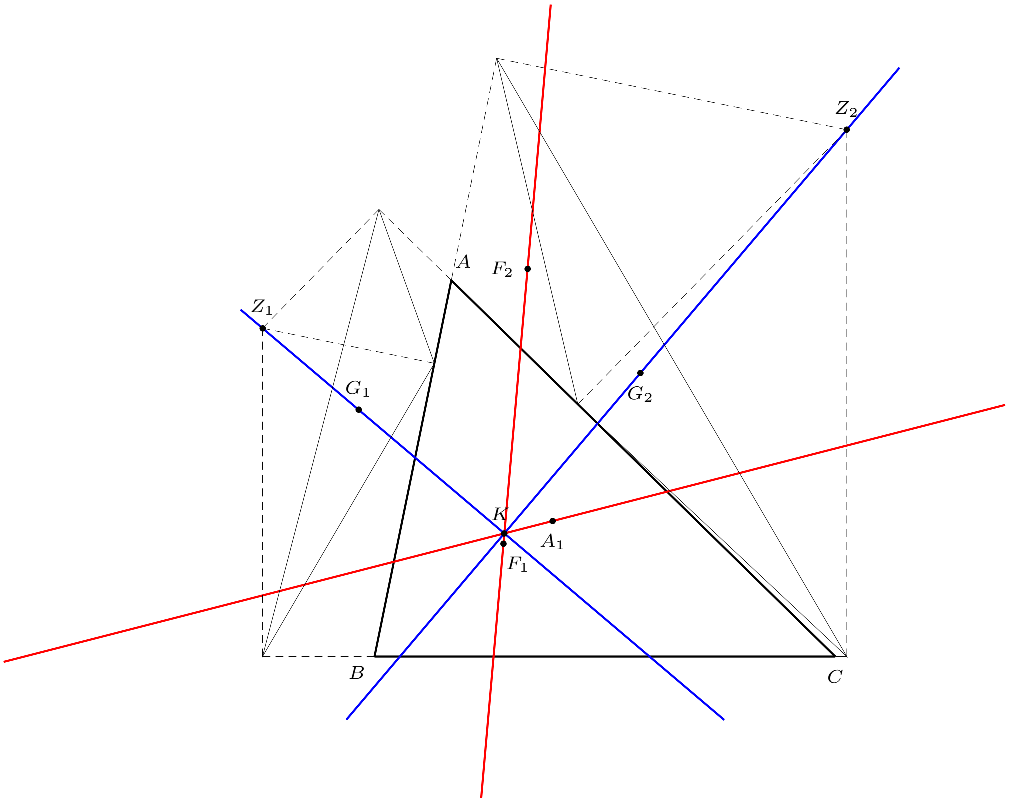

Figure 5

References

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000, http://cedar.evansville.edu/˜ck6/encyclopedia/.

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