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# Pedal Triangles and Their Shadows

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**Abstract**. The pedal triangle of a point $P$ with respect to a given triangle $ABC$ casts equal shadows on the side lines of $ABC$ if and only if $P$ is the internal center of similitude of the circumcircle and the incircle of triangle $ABC$ or the external center of the circumcircle with one of the excircles. We determine the common length of the equal shadows. More generally, we construct the point the shadows of whose pedal triangle are proportional to given $p$, $q$, $r$. Many interesting special cases are considered.

## 1. Shadows of pedal triangle

Let $P$ be a point in the plane of triangle $ABC$, and $A'B'C'$ its pedal triangle, *i.e.*, $A'$, $B'$, $C'$ are the pedals (orthogonal projections) of $A$, $B$, $C$ on the side lines $BC$, $CA$, $AB$ respectively. If $B_a$ and $C_a$ are the pedals of $B'$ and $C'$ on $BC$, we call the segment $B_aC_a$ the *shadow* of $B'C'$ on $BC$. The shadows of $C'A'$ and $A'B'$ are segments $C_bA_b$ and $A_cB_c$ analogously defined on the lines $CA$ and $AB$. See Figure 1.

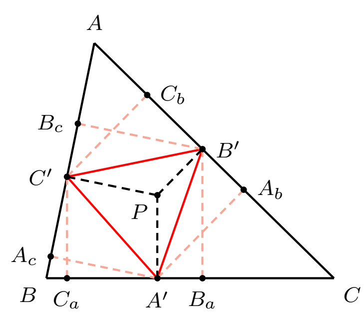


Figure 1

In terms of the *actual* normal coordinates $x$, $y$, $z$ of $P$ with respect to $ABC$,[1] the length of the shadow $C_aB_a$ can be easily determined:

$$C_aB_a = CaA' + A'B_a = z\sin B + y\sin C. \tag{1}$$

In Figure 1, we have shown $P$ as interior point of triangle $ABC$. For generic positions of $P$, we regard $C_aB_a$ as a directed segment so that its length given by (1) is signed. Similarly, the shadows of $C'A'$ and $A'B'$ on the respective side lines have signed lengths $x\sin C + z\sin A$ and $y\sin A + x\sin B$.

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[1]Traditionally, normal coordinates are called trilinear coordinates. Here, we follow the usage of the old French term *coordonnées normales* in F.G.-M. [1], which is more suggestive. The actual normal (trilinear) coordinates of a point are the *signed* distances from the point to the three side lines.

**Theorem 1.** *The three shadows of the pedal triangle of $P$ on the side lines are equal if and only if $P$ is the internal center of similitude of the circumcircle and the incircle of triangle $ABC$, or the external center of similitude of the circumcircle and one of the excircles.*

*Proof.* These three shadows are equal if and only if

$$\epsilon_1(y\sin C+z\sin B)=\epsilon_2(z\sin A+x\sin C)=\epsilon_3(x\sin B+y\sin A)$$

for an appropriate choice of signs $\epsilon_1,\epsilon_2,\epsilon_3=\pm1$ subject to the convention

$(\star)$ *at most one of* $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ *is negative.*

It follows that

$$\begin{array}{rcrcrcl} x\epsilon_2\sin C & - & y\epsilon_1\sin C & + & z(\epsilon_2\sin A-\epsilon_1\sin B) & = & 0,\\ x(\epsilon_3\sin B-\epsilon_2\sin C) & + & y\epsilon_3\sin A & - & z\epsilon_2\sin A & = & 0.\end{array}$$

Replacing, by the law of sines, $\sin A$, $\sin B$, $\sin C$ by the side lengths $a$, $b$, $c$ respectively, we have

$$\begin{aligned} x:y:z &= \begin{vmatrix}-\epsilon_1c & \epsilon_2a-\epsilon_1b\\ \epsilon_3a & -\epsilon_2a\end{vmatrix}:-\begin{vmatrix}\epsilon_2c & \epsilon_2a-\epsilon_1b\\ \epsilon_3b-\epsilon_2c & -\epsilon_2a\end{vmatrix}:\begin{vmatrix}\epsilon_2c & -\epsilon_1c\\ \epsilon_3b-\epsilon_2c & \epsilon_3a\end{vmatrix}\\ &= a(\epsilon_3\epsilon_1b+\epsilon_1\epsilon_2c-\epsilon_2\epsilon_3a):b(\epsilon_1\epsilon_2c+\epsilon_2\epsilon_3a-\epsilon_3\epsilon_1b):c(\epsilon_2\epsilon_3a+\epsilon_3\epsilon_1b-\epsilon_1\epsilon_2c)\\ &= a(\epsilon_2b+\epsilon_3c-\epsilon_1a):b(\epsilon_3c+\epsilon_1a-\epsilon_2b):c(\epsilon_1a+\epsilon_2b-\epsilon_3c). \qquad (2)\end{aligned}$$

If $\epsilon_1=\epsilon_2=\epsilon_3=1$, this is the point $X_{55}$ in [4], the internal center of similitude of the circumcircle and the incircle. We denote this point by $T$. See Figure 2A. We show that if one of $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ is negative, then $P$ is the external center of similitude of the circumcircle and one of the excircles.

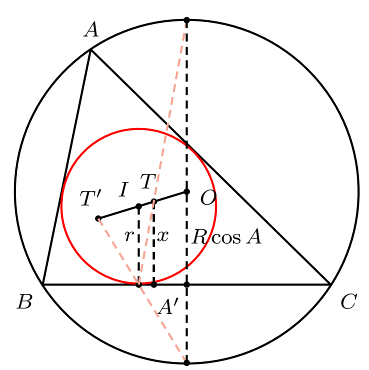


Figure 2A

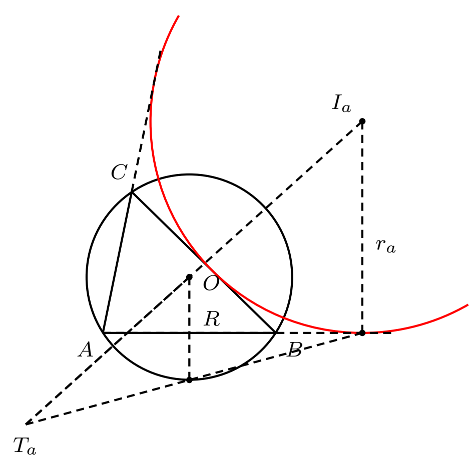


Figure 2B

Let $R$ denote the circumradius, $s$ the semiperimeter, and $r_a$ the radius of the $A$-excircle. The actual normal coordinates of the circumcenter are $R\cos A$, $R\cos B$,

$R\cos C$, while those of the excenter $I_a$ are $-r_a, r_a, r_a$. See Figure 2B. The external center of similitude of the two circles is the point $T_a$ dividing $I_aO$ in the ratio $I_aT_a : T_aO = r_a : -R$. As such, it is the point $\frac{1}{r_a-R}(r_a\cdot O - R\cdot I_a)$, and has normal coordinates

$$\begin{aligned}
& -(1+\cos A) : 1-\cos B : 1-\cos C \\
= \quad & -\cos^2\frac{A}{2} : \sin^2\frac{B}{2} : \sin^2\frac{C}{2} \\
= \quad & -a(a+b+c) : b(a+b-c) : c(c+a-b).
\end{aligned}$$

This coincides with the point given by (2) for $\epsilon_1 = -1$, $\epsilon_2 = \epsilon_3 = 1$. The cases for other choices of signs are similar, leading to the external centers of similitude with the other two excircles. $\square$

*Remark.* With these coordinates, we easily determine the common length of the equal shadows in each case. For the point $T$, this common length is

$$\begin{aligned}
y\sin C + z\sin B &= \frac{Rr}{R+r}((1+\cos B)\sin C + (1+\cos C)\sin B) \\
&= \frac{Rr}{R+r}(\sin A + \sin B + \sin C) \\
&= \frac{1}{R+r}\cdot\frac{1}{2}(a+b+c)r \\
&= \frac{\triangle}{R+r},
\end{aligned}$$

where $\triangle$ denotes the area of triangle $ABC$. For $T_a$, the common length of the equal shadows is $\left|\frac{\triangle}{r_a-R}\right|$; similarly for the other two external centers of similitudes.

## 2. Pedal triangles with shadows in given proportions

If the signed lengths of the shadows of the sides of the pedal triangle of $P$ (with normal coordinates $(x : y : z)$) are proportional to three given quantities $p$, $q$, $r$, then

$$\frac{cy+bz}{p} = \frac{az+cx}{q} = \frac{bx+ay}{r}.$$

From these, we easily obtain the normal of coordinates of $P$:

$$(a(-ap+bq+cr) : b(ap-bq+cr) : c(ap+bq-cr)). \tag{3}$$

This follows from a more general result, which we record for later use.

**Lemma 2.** *The solution of*

$$f_1x + g_1y + h_1z = f_2x + g_2y + h_2z = f_3x + g_3y + h_3z$$

*is*

$$x : y : z = \begin{vmatrix} 1 & g_1 & h_1 \\ 1 & g_2 & h_2 \\ 1 & g_3 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & 1 & h_1 \\ f_2 & 1 & h_2 \\ f_3 & 1 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix}.$$

*Proof.* since there are two linear equations in three indeterminates, solution is unique up to a proportionality constant. To verify that this is the correct solution, note that for $i = 1, 2, 3$, substitution into the $i$-th linear form gives

$$-\begin{vmatrix} 0 & f_i & g_i & h_i \\ 1 & f_1 & g_1 & h_1 \\ 1 & f_2 & g_2 & h_2 \\ 1 & f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & f_1 & g_1 & h_1 \\ 1 & f_2 & g_2 & h_2 \\ 1 & f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$$

up to a constant. □

**Proposition 3.** *The point the shadows of whose pedal triangle are in the ratio $p : q : r$ is the perspector of the cevian triangle of the point with normal coordinates $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ and the tangential triangle of $ABC$.*

*Proof.* If $Q$ is the point with normal coordinates $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$, then $P$, with coordinates given by (3), is the $Q$-Ceva conjugate of the symmedian point $K = (a : b : c)$. See [3, p.57]. □

If we assume $p$, $q$, $r$ positive, there are four points satisfying

$$\frac{cy + bz}{\epsilon_1 p} = \frac{az + cx}{\epsilon_2 q} = \frac{bx + ay}{\epsilon_3 r},$$

for signs $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ satisfying ($\star$). Along with $P$ given by (3), there are

$$\begin{aligned} P_a &= (-a(ap + bq + cr) : b(ap + bq - cr) : c(ap - bq + cr)), \\ P_b &= (a(-ap + bq + cr) : -b(ap + bq + cr) : c(-ap + bq + cr)), \\ P_c &= (a(ap - bq + cr) : b(-ap + bq + cr) : -c(ap + bq + cr)). \end{aligned}$$

While it is clear that $P_aP_bP_c$ is perspective with $ABC$ at

$$\left(\frac{a}{-ap + bq + cr} : \frac{b}{ap - bq + cr} : \frac{c}{ap + bq - cr}\right),$$

the following observation is more interesting and useful in the construction of these points from $P$.

**Proposition 4.** *$P_aP_bP_c$ is the anticevian triangle of $P$ with respect to the tangential triangle of $ABC$.*

*Proof.* The vertices of the tangential triangle are

$$A' = (-a : b : c), \qquad B' = (a : -b : c), \qquad C' = (a : b : -c).$$

From

$$\begin{aligned} & (a(-ap + bq + cr),\ b(ap - bq + cr),\ c(ap + bq - cr)) \\ = \;& ap(-a, b, c) + (a(bq + cr),\ -b(bq - cr),\ c(bq - cr)), \end{aligned}$$

and

$$\begin{aligned} & (-a(ap + bq + cr),\ b(ap + bq - cr),\ c(ap - bq + cr)) \\ = \;& ap(-a, b, c) - (a(bq + cr),\ -b(bq - cr),\ c(bq - cr)), \end{aligned}$$

we conclude that $P$ and $P_a$ divide $A'$ and $A'' = (a(bq+r) : -b(bq-cr) : c(bq-cr))$ harmonically. But since

$$(a(bq+cr),\ -b(bq-cr),\ c(bq-cr)) = bq(a,-b,c) + cr(a,b,-c),$$

the point $A''$ is on the line $B'C'$. The cases for $P_b$ and $P_c$ are similar, showing that triangle $P_aP_bP_c$ is the anticevian triangle of $P$ in the tangential triangle. □

## 3. Examples

3.1. *Shadows proportional to side lengths.* If $p : q : r = a : b : c$, then $P$ is the circumcenter $O$. The pedal triangle of $O$ being the medial triangle, the lengths of the shadows are halves of the side lengths. Since the circumcenter is the incenter or one of the excenters of the tangential triangle (according as the triangle is acute- or obtuse-angled), the four points in question are the circumcenter and the excenters of the tangential triangle. [2]

3.2. *Shadows proportional to altitudes.* If $p : q : r = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, then $P$ is the symmedian point $K = (a : b : c)$. [3] Since $K$ is the Gergonne point of the tangential triangle, the other three points, with normal coordinates $(3a : -b : -c)$, $(-a : 3b : -c)$, and $(-a : -b : 3c)$, are the Gergonne points of the excircles of the tangential triangle. These are also the cases when the shadows are inversely proportional to the distances from $P$ to the side lines, or, equivalently, when the triangles $PB_aC_a$, $PC_aB_a$ and $PA_cB_c$ have equal areas. [4]

3.3. *Shadows inversely proportional to exradii.* If $p : q : r = \frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c} = b+c-a : c+a-b : a+b-c$, then $P$ is the point with normal coordinates $(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c}) = (ar_a : br_b : cr_c)$. This is the *external* center of similitude of the circumcircle and the incircle, which we denote by $T'$. See Figure 2A. This point appears as $X_{56}$ in [4]. The other three points are the *internal* centers of similitude of the circumcircle and the three excircles.

3.4. *Shadows proportional to exradii.* If $p : q : r = r_a : r_b : r_c = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2}$, then $P$ has normal coordinates

$$\begin{aligned}
&a(b\tan\frac{B}{2} + c\tan\frac{C}{2} - a\tan\frac{A}{2}) : b(c\tan\frac{C}{2} + a\tan\frac{A}{2} - b\tan\frac{B}{2}) : c(a\tan\frac{A}{2} + b\tan\frac{B}{2} - c\tan\frac{C}{2})\\
\sim&2a(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2}) : 2b(\sin^2\frac{C}{2} + \sin^2\frac{A}{2} - \sin^2\frac{B}{2}) : 2c(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2})\\
\sim&a(1+\cos A - \cos B - \cos C) : b(1+\cos B - \cos C - \cos A) : c(1+\cos C - \cos A - \cos B).
\end{aligned} \tag{4}$$

[2]If $ABC$ is right-angled, the tangential triangle degenerates into a pair of parallel lines, and there is only one finite excenter.

[3]More generally, if $p : q : r = a^n : b^n : c^n$, then the normal coordinates of $P$ are

$$(a(b^{n+1} + c^{n+1} - a^{n+1}) : b(c^{n+1} + a^{n+1} - b^{n+1}) : c(a^{n+1} + b^{n+1} - c^{n+1})).$$

[4]For signs $\epsilon_1, \epsilon_2, \epsilon_3$ satisfying $(\star)$, the equations $\epsilon_1 x(cy+bz) = \epsilon_2 y(az+cx) = \epsilon_3 z(bx+ay)$ can be solved for $yz : zx : xy$ by an application of Lemma 2. From this it follows that $x : y : z = (\epsilon_2+\epsilon_3-\epsilon_1)a : (\epsilon_3+\epsilon_1-\epsilon_2)b : (\epsilon_1+\epsilon_2-\epsilon_3)c$.

This is the point $X_{198}$ of [4]. It can be constructed, according to Proposition 3, from the point with normal coordinates $(\frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c}) = (s-a : s-b : s-c)$, the Mittenpunkt. [5]

## 4. A synthesis

The five triangle centers we obtained with special properties of the shadows of their pedal triangles, namely, $O$, $K$, $T$, $T'$, and the point $P$ in §3.4, can be organized together in a very simple way. We take a closer look at the coordinates of $P$ given in (4) above. Since

$$1 - \cos A + \cos B + \cos C = 2 - 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2 - \frac{r_a}{R},$$

the normal coordinates of $P$ can be rewritten as

$$(a(2R - r_a) : b(2R - r_b) : c(2R - r_c)).$$

These coordinates indicate that $P$ lies on the line joining the symmedian point $K(a : b : c)$ to the point $(ar_a : br_b : cr_c)$, the point $T'$ in §3.3, with division ratio

$$\begin{aligned} T'P : PK &= 2R(a^2+b^2+c^2) : -(a^2r_a + b^2r_b + c^2r_c) \\ &= R(a^2+b^2+c^2) : -2(R-r)s^2. \end{aligned} \tag{5}$$

To justify this last expression, we compute in two ways the distance from $T'$ to the line $BC$, and obtain

$$\frac{2\triangle}{a^2r_a + b^2r_b + c^2r_c} \cdot ar_a = \frac{Rr}{R-r}(1 - \cos A).$$

From this,

$$\begin{aligned} a^2r_a + b^2r_b + c^2r_c &= \frac{2\triangle(R-r)}{Rr} \cdot \frac{ar_a}{1-\cos A} \\ &= \frac{2\triangle(R-r)}{Rr} \cdot \frac{4R\sin\frac{A}{2}\cos\frac{A}{2} \cdot s\tan\frac{A}{2}}{2\sin^2\frac{A}{2}} \\ &= 4(R-r)s^2. \end{aligned}$$

This justifies (5) above.

Consider the intersection $X$ of the line $TP$ with $OK$. See Figure 3. Applying Menelaus' theorem to triangle $OKT'$ with transversal $TXP$, we have

$$\frac{OX}{XK} = -\frac{OT}{TT'} \cdot \frac{T'P}{PK} = \frac{R-r}{2r} \cdot \frac{R(a^2+b^2+c^2)}{2(R-r)s^2} = \frac{R(a^2+b^2+c^2)}{4\triangle s}.$$

This expression has an interesting interpretation. The point $X$ being on the line $OK$, it is the isogonal conjugate of a point on the Kiepert hyperbola. Every point on this hyperbola is the perspector of the apexes of similar isosceles triangles constructed on the sides of $ABC$. If this angle is taken to be $\arctan\frac{s}{R}$, and the

[5]This appears as $X_9$ in [4], and can be constructed as the perspector of the excentral triangle and the medial triangle, *i.e.*, the intersection of the three lines each joining an excenter to the midpoint of the corresponding side of triangle $ABC$.

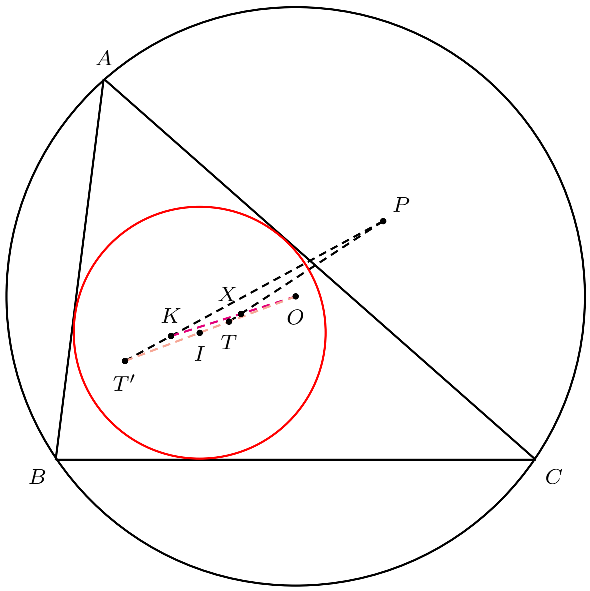


Figure 3

isosceles triangles constructed externally on the sides of triangle $ABC$, then the isogonal conjugate of the perspector is precisely the point $X$.

This therefore furnishes a construction for the point $P$.[6]

## 5. Two more examples

5.1. *Shadows of pedal triangle proportional to distances from circumcenter to side lines.* The point $P$ is the perspector of the tangential triangle and the cevian triangle of $(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C})$, which is the orthic triangle of $ABC$. The two triangles are indeed homothetic at the Gob perspector on the Euler line. See [2, pp.259–260]. It has normal coordinates $(a\tan A : b\tan B : c\tan C)$, and appears as $X_{25}$ in [4].

5.2. *Shadows of pedal triangles proportional to distances from orthocenter to side lines.* In this case, $P$ is the perspector of the tangential triangle and the cevian triangle of the circumcenter. This is the point with normal coordinates

$$(a(-\tan A+\tan B+\tan C) : b(\tan A-\tan B+\tan C) : c(\tan A+\tan B-\tan C)),$$

and is the centroid of the tangential triangle. It appears as $X_{154}$ in [4]. The other three points with the same property are the vertices of the anticomplementary triangle of the tangential triangle.

[6] The same $P$ can also be constructed as the intersection of $KT'$ and the line joining the incenter to $Y$ on $OK$, which is the isogonal conjugate of the perspector (on the Kiepert hyperbola) of apexes of similar isosceles triangles with base angles $\arctan\frac{s}{2R}$ constructed externally on the sides of $ABC$.

## 6. The midpoints of shadows as pedals

The midpoints of the shadows of the pedal triangle of $P = (x : y : z)$ are the pedals of the point

$$P' = (x+y\cos C+z\cos B : \ y+z\cos A+x\cos C : \ z+x\cos B+y\cos A) \quad (6)$$

in normal coordinates. This is equivalent to the concurrency of the perpendiculars from the midpoints of the sides of the pedal triangle of $P$ to the corresponding sides of $ABC$.[7] See Figure 4.

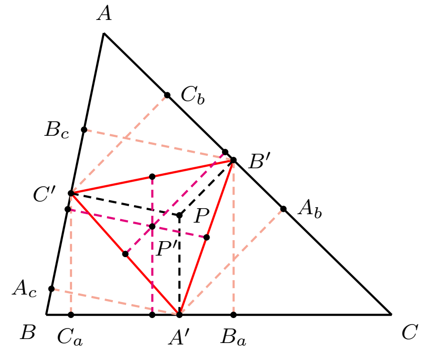


Figure 4

If $P$ is the symmedian point, with normal coordinates $(\sin A : \sin B : \sin C)$, it is easy to see that $P'$ is the same symmedian point.

**Proposition 5.** *There are exactly four points for each of which the midpoints of the sides of the pedal triangle are equidistant from the corresponding sides of $ABC$.*

*Proof.* The midpoints of the sides of the pedal triangle have *signed* distances

$$x+\frac{1}{2}(y\cos C+z\cos B), \quad y+\frac{1}{2}(z\cos A+x\cos C), \quad z+\frac{1}{2}(x\cos B+y\cos A)$$

from the respective sides of $ABC$. The segments joining the midpoints of the sides and their shadows are equal in length if and only if

$$\epsilon_1(2x+y\cos C+z\cos B) = \epsilon_2(2y+z\cos A+x\cos C) = \epsilon_3(2z+x\cos B+y\cos A)$$

for $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ satisfying $(\star)$. From these, we obtain the four points.

For $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this gives the point

$$\begin{aligned} M = ((2-\cos A)(2+\cos A-\cos B-\cos C) \\ :(2-\cos B)(2+\cos B-\cos C+\cos A) \\ :(2-\cos C)(2+\cos C-\cos A+\cos B)) \end{aligned}$$

in normal coordinates, which can be constructed as the incenter-Ceva conjugate of

$$Q = (2-\cos A : 2-\cos B : 2-\cos C),$$

[7] If $x$, $y$, $z$ are the actual normal coordinates of $P$, then those of $P'$ are halves of those given in (6) above, and $P'$ is $\frac{x}{2}$, $\frac{y}{2}$, and $\frac{z}{2}$ below the midpoints of the respective sides of the pedal triangle.

See [3, p.57]. This point $Q$ divides the segments $OI$ externally in the ratio $OQ : QI = 2R : -r$. See Figures 5A and 5B.

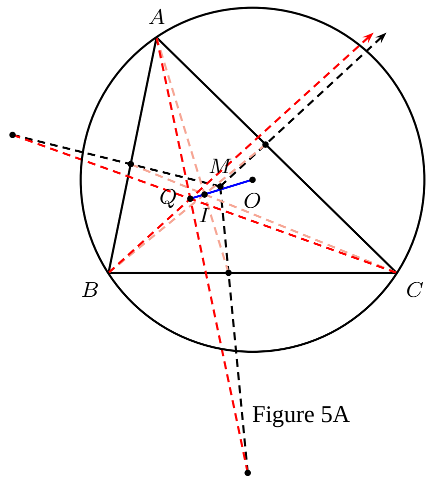


Figure 5A

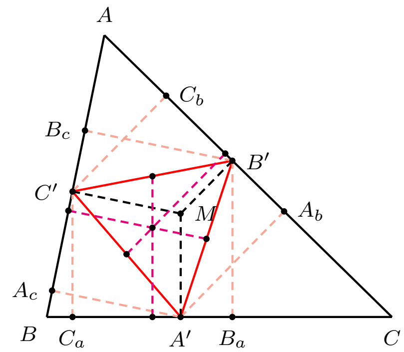


Figure 5B

There are three other points obtained by choosing one negative sign among $\epsilon_1$, $\epsilon_2$, $\epsilon_3$. These are

$$\begin{aligned} M_a = (&-(2-\cos A)(2+\cos A+\cos B+\cos C)\\ &:(2+\cos B)(2-\cos A-\cos B+\cos C)\\ &:(2+\cos C)(2-\cos A+\cos B-\cos C)),\end{aligned}$$

and $M_b$, $M_c$ whose coordinates can be written down by appropriately changing signs. It is clear that $M_aM_bM_c$ and triangle $ABC$ are perspective at

$$M' = \left(\frac{2+\cos A}{2+\cos A-\cos B-\cos C} : \frac{2+\cos B}{2-\cos A+\cos B-\cos C} : \frac{2+\cos C}{2-\cos A-\cos B+\cos C}\right).$$

□

The triangle centers $Q$, $M$, and $M'$ in the present section apparently are not in [4].

## Appendix: Pedal triangles of a given shape

The side lengths of the pedal triangle of $P$ are given by $AP \cdot \sin A$, $BP \cdot \sin B$, and $CP \cdot \sin C$. [2, p.136]. This is similar to one with side lengths $p : q : r$ if and only if the *tripolar* coordinates of $P$ are

$$AP : BP : CP = \frac{p}{a} : \frac{q}{b} : \frac{r}{c}.$$

In general, there are two such points, which are common to the three generalized Apollonian circles associated with the point $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ in *normal* coordinates. See, for example, [5]. In the case of equilateral triangles, these are the isodynamic points.

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