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Perspective Poristic Triangles

Edward Brisse

Abstract. This paper answers a question of Yiu: given a triangle ABC, to construct and enumerate the triangles which share the same circumcircle and incircle and are perspective with ABC. We show that there are exactly three such triangles, each easily constructible using ruler and compass.

1. Introduction

Given a triangle ABC with its circumcircle $O(R)$ and incircle $I(r)$, the famous Poncelet - Steiner porism affirms that there is a continuous family of triangles with the same circumcircle and incircle [1, p.86]. Every such triangle can be constructed by choosing an arbitrary point A' on the circle (O), drawing the two tangents to (I), and extending them to intersect (O) again at B' and C'. Yiu [3] has raised the enumeration and construction problems of poristic triangles perspective with triangle ABC, namely, those poristic triangles $A'B'C'$ with the lines AA', BB', CC' intersecting at a common point. We give a complete solution to these problems in terms of the limit points of the coaxial system of circles generated by the circumcircle and the incircle.

Theorem 1. *The only poristic triangles perspective with ABC are:*

(1) *the reflection of ABC in the line OI, the perspector being the infinite point on a line perpendicular to OI,*
(2) *the circumcevian triangles of the two limit points of the coaxial system generated by the circumcircle and the incircle.*

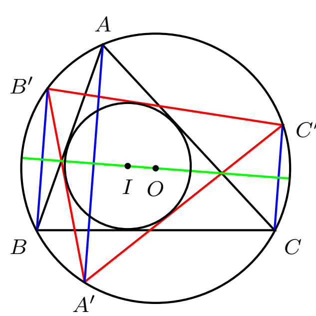

Figure 1

In (1), the lines AA', BB', CC' are all perpendicular to the line OI. See Figure 1. The perspector is the infinite point on a line perpendicular to OI. One such line

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is the trilinear polar of the incenter $I = (a : b : c)$, with equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$$

in homogeneous barycentric coordinates. The perspector is therefore the point $(a(b-c) : b(c-a) : c(a-b))$. We explain in §§2, 3 the construction of the two triangles in (2), which are symmetric with respect to the line OI. See Figure 2. In §4 we justify that these three are the only poristic triangles perspective with ABC.

2. Poristic triangles from an involution in the upper half-plane

An easy description of the poristic triangles in Theorem 1(2) is that these are the circumcevian triangles of the common poles of the circumcircle and the incircle. There are two such points; each of these has the same line as the polar with respect the circumcircle and the incircle. These common poles are symmetric with respect

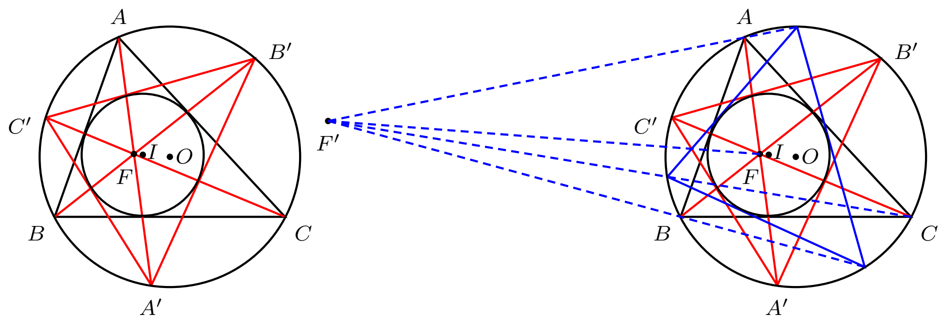

Figure 2

to the radical axis of the circles (O) and (I), and are indeed the *limit points* of the coaxial system of circles generated by (O) and (I).[1] This is best explained by the introduction of an involution of the upper half-plane. Let $a > 0$ be a fixed real number. Consider in the upper half-plane $\mathcal{R}^2_+ := \{(x, y) : y > 0\}$ a family of circles

$$\mathcal{C}_b : \qquad x^2 + y^2 - 2by + a^2 = 0, \qquad b \geq a.$$

Each circle $\mathcal{C}_b$ has center $(0, b)$ and radius $\sqrt{b^2 - a^2}$. See Figure 3. Every point in $\mathcal{R}^2_+$ lies on a unique circle $\mathcal{C}_b$ in this family. Specifically, if

$$b(x, y) = \frac{x^2 + y^2 + a^2}{2y},$$

the point (x, y) lies on the circle $\mathcal{C}_{b(x,y)}$. The circle $\mathcal{C}_a$ consists of the single point $F = (0, a)$. We call this the limit point of the family of circles. Every pair of circles in this family has the x-axis as radical axis. By reflecting the system of circles about the x-axis, we obtain a complete coaxial system of circles. The reflection of F, namely, the point $F' = (0, -a)$, is the other limit point of this system. Every circle through F and F' is orthogonal to every circle $\mathcal{C}_b$.

[1]The common polar of each one of these points with respect to the two circles passes through the other.

Consider a line through the limiting point F, with slope m, and therefore equation $y = mx + a$. This line intersects the circle $\mathcal{C}_b$ at points whose y-coordinates are the roots of the quadratic equation

$$(1 + m^2)y^2 - 2(a + bm^2)y + a^2(1 + m^2) = 0.$$

Note that the two roots multiply to a^2. Thus, if one of the intersections is (x, y), then the other intersection is $(-\frac{ax}{y}, \frac{a^2}{y})$. See Figure 4. This defines an involution on the upper half plane:

$$P^* = (-\frac{ax}{y}, \frac{a^2}{y}) \quad \text{for} \quad P = (x, y).$$

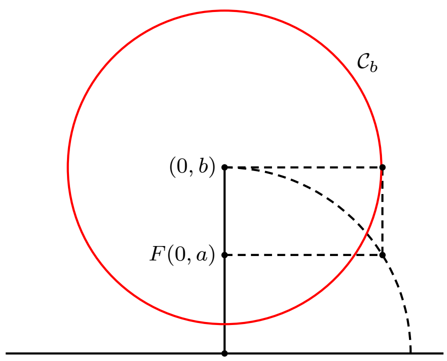

Figure 3

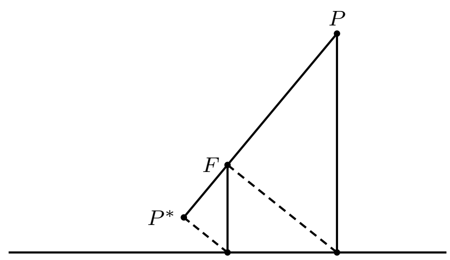

Figure 4

Proposition 2. *(1) $P^{**} = P$.*
(2) P and P^* belong to the same circle in the family $\mathcal{C}_b$. In other words, if P lies on the circle $\mathcal{C}_b$, then the line FP intersects the same circle again at P^*.
(3) The line PF' intersects the circle $\mathcal{C}_b$ at the reflection of P^* in the y-axis.

Proof. (1) is trivial. (2) follows from $b(P) = b(P^*)$. For (3), the intersection is the point $(\frac{ax}{y}, \frac{a^2}{y})$. $\square$

Lemma 3. *Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points on the same circle $\mathcal{C}_b$. The segment AB is tangent to a circle $\mathcal{C}_{b'}$ at the point whose y-coordinate is $\sqrt{y_1 y_2}$.*

Proof. This is clear if $y_1 = y_2$. In the generic case, extend AB to intersect the x-axis at a point C. The segment AB is tangent to a circle $\mathcal{C}_{b'}$ at a point P such that $CP = CF$. It follows that $CP^2 = CF^2 = CA \cdot CB$. Since C is on the x-axis, this relation gives $y^2 = y_1 y_2$ for the y-coordinate of P. $\square$

Theorem 4. *If a chord AB of $\mathcal{C}_b$ is tangent to $\mathcal{C}_{b'}$ at P, then the chord A^*B^* is tangent to the same circle $\mathcal{C}_{b'}$ at P^*.*

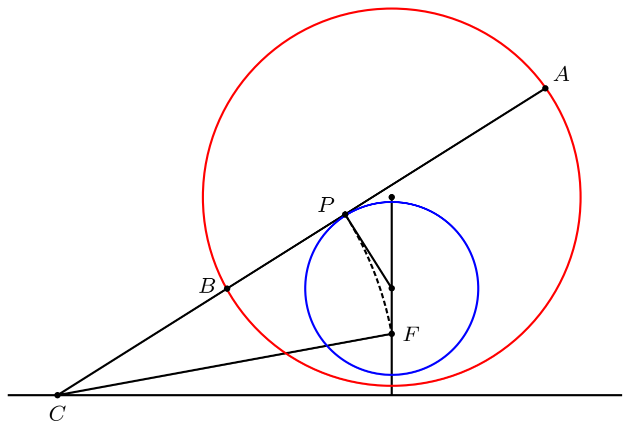

Figure 5

Proof. That P and P^* lie on the same circle is clear from Proposition 2(1). It remains to show that P^* is the correct point of tangency. This follows from noting that the y-coordinate of P^*, being $\frac{a^2}{\sqrt{y_1 y_2}}$, is the geometric mean of those of A^* and B^*. □

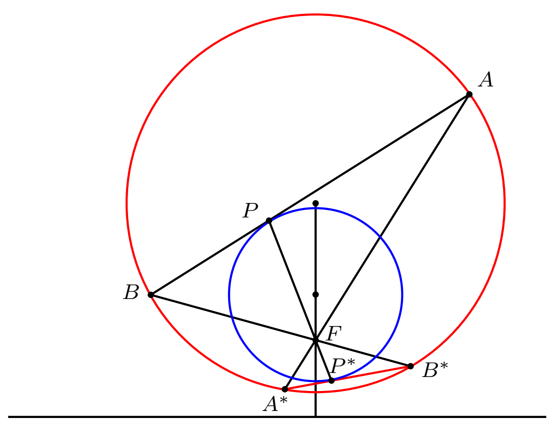

Figure 6

Consider the circumcircle and incircle of triangle ABC. These two circles generate a coaxial system with limit points F and F'.

Corollary 5. *The triangle $A^*B^*C^*$ has $I(r)$ as incircle, and is perspective with ABC at F.*

Corollary 6. *The reflection of the triangle $A^*B^*C^*$ in the line OI also has $I(r)$ as incircle, and is perspective with ABC at the point F'.*

Proof. This follows from Proposition 2 (3). □

It remains to construct the two limit points F and F', and the construction of the two triangles in Theorem 1(2) would be complete.

Proposition 7. *Let XY be the diameter of the circumcircle through the incenter I. If the tangents to the incircle from these two points are XP, XQ', YQ, and YP' such that P and Q are on the same side of OI, then PP' intersects OI at F (so does QQ'), and PQ intersects OI at F' (so does $P'Q'$).*

Proof. This follows from Theorem 4 by observing that Y = X*. □

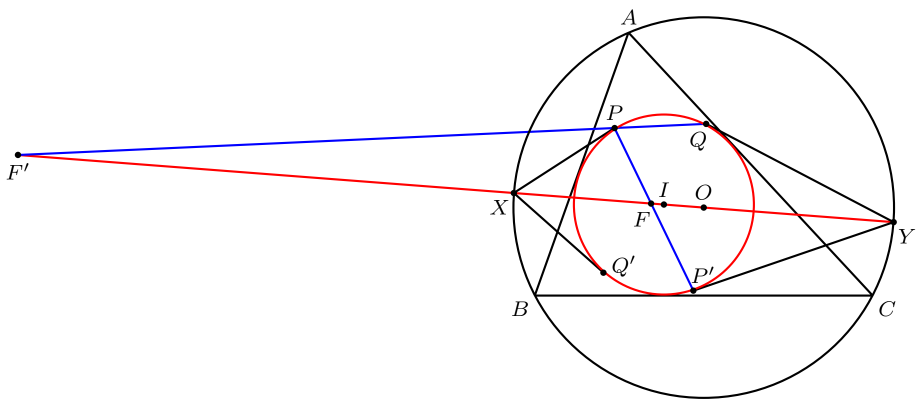

Figure 7

3. Enumeration of perspective poristic triangles

In this section, we show that the poristic triangles constructed in the preceding sections are the only ones perspective with ABC. To do this, we adopt a slightly different viewpoint, by searching for circumcevian triangles which share the same incircle with ABC. We work with homogeneous barycentric coordinates. Recall that if a, b, c are the lengths of the sides BC, CA, AB respectively, then the circumcircle has equation

$$a^2yz + b^2zx + c^2xy = 0,$$

and the incircle has equation

$$\begin{aligned}&(s-a)^2x^2 + (s-b)^2y^2 + (s-c)^2z^2 - 2(s-b)(s-c)yz\\&-2(s-c)(s-a)zx - 2(s-a)(s-b)xy = 0,\end{aligned}$$

where $s = \frac{1}{2}(a+b+c)$.

We begin with a lemma.

Lemma 8. *The tangents from a point $(u : v : w)$ on the circumcircle (O) to the incircle (I) intersect the circumcircle again at two points on the line*

$$\frac{(s-a)u}{a^2}x + \frac{(s-b)v}{b^2}y + \frac{(s-c)w}{c^2}z = 0.$$

Remark: This line is tangent to the incircle at the point

$$\left(\frac{a^4}{(s-a)u^2} \;:\; \frac{b^4}{(s-b)v^2} \;:\; \frac{c^4}{(s-c)w^2}\right).$$

Given a point $P = (u : v : w)$ in homogeneous barycentric coordinates, the circumcevian triangle $A'B'C'$ is formed by the *second* intersections of the lines AP, BP, CP with the circumcircle. These have coordinates

$$A' = (\frac{-a^2vw}{b^2w+c^2v} : v : w), \quad B' = (u : \frac{-b^2wu}{c^2u+a^2w} : w), \quad C' = (u : v : \frac{-c^2uv}{a^2v+b^2u}).$$

Applying Lemma 8 to the point A', we obtain the equation of the line $B'C'$ as

$$\frac{-(s-a)vw}{b^2w+c^2v}x + \frac{(s-b)v}{b^2}y + \frac{(s-c)w}{c^2}z = 0.$$

Since this line contains the points B' and C', we have

$$-\frac{(s-a)uvw}{b^2w+c^2v} - \frac{(s-b)uvw}{c^2u+a^2w} + \frac{(s-c)w^2}{c^2} = 0, \tag{1}$$

$$-\frac{(s-a)uvw}{b^2w+c^2v} + \frac{(s-b)v^2}{b^2} - \frac{(s-c)uvw}{b^2u+a^2v} = 0. \tag{2}$$

The difference of these two equations gives

$$\frac{a^2vw+b^2wu+c^2uv}{b^2c^2(b^2u+a^2v)(c^2u+a^2w)} \cdot f = 0, \tag{3}$$

where

$$f = -b^2c^2(s-b)uv + b^2c^2(s-c)wu - c^2a^2(s-b)v^2 + a^2b^2(s-c)w^2.$$

If $a^2vw + b^2wu + c^2uv = 0$, the point $(u : v : w)$ is on the circumcircle, and both equations (1) and (2) reduce to

$$\frac{s-a}{a^2}u^2 + \frac{s-b}{b^2}v^2 + \frac{s-c}{c^2}w^2 = 0,$$

clearly admitting no real solutions. On the other hand, setting the quadratic factor f in (3) to 0, we obtain

$$u = \frac{-a^2}{b^2c^2} \cdot \frac{c^2(s-b)v^2 - b^2(s-c)w^2}{(s-b)v-(s-c)w}.$$

Substitution into equation (1) gives

$$\frac{vw(c(a-b)v-b(c-a)w)}{b^2c^2(c^2v^2-b^2w^2)(v(s-b)-w(s-c))} \cdot g = 0, \tag{4}$$

where

$$\begin{aligned} g \;=\;& c^3(s-b)(a^2+b^2-c(a+b))v^2 + b^3(s-c)(c^2+a^2-b(c+a))w^2 \\ &+2bc(s-b)(s-c)(b^2+c^2-a(b+c))vw. \end{aligned}$$

There are two possibilities.

(i) If $c(a-b)v - b(c-a)w = 0$, we obtain $v : w = b(c-a) : c(a-b)$, and consequently, $u : v : w = a(b-c) : b(c-a) : c(a-b)$. This is clearly an infinite point, the one on the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$, the trilinear polar of the incenter. This line is perpendicular to the line OI. This therefore leads to the triangle in Theorem 1(1).

(ii) Setting the quadratic factor g in (4) to 0 necessarily leads to the two triangles constructed in §2. The corresponding perspectors are the two limit points of the coaxial system generated by the circumcircle and the incircle.

4. Coordinates

The line OI has equation

$$\frac{(b-c)(s-a)}{a}x + \frac{(c-a)(s-b)}{b}y + \frac{(a-b)(s-c)}{c}z = 0.$$

The radical axis of the two circles is the line

$$(s-a)^2x + (s-b)^2y + (s-c)^2z = 0.$$

These two lines intersect at the point

$$\left(\frac{a(a^2(b+c) - 2a(b^2 - bc + c^2) + (b-c)^2(b+c))}{b+c-a} : \cdots : \cdots\right),$$

where the second and third coordinates are obtained from the first by cyclic permutations of a, b, c. This point is not found in [2].

The coordinates of the common poles F and F' are

$$(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)) + t(a : b : c)$$

where

$$t = \frac{1}{2}\left(-2abc + \sum_{\text{cyclic}}(a^3 - bc(b+c))\right) \pm 2\triangle\sqrt{2ab+2bc+2ca-a^2-b^2-c^2}, \tag{5}$$

and $\triangle$ = area of triangle ABC. This means that the points F and F' divide harmonically the segment joining the incenter $I(a : b : c)$ to the point whose homogeneous barycentric coordinates are

$$\begin{aligned} & (a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)) \\ + \quad & \frac{1}{2}\left(-2abc + \sum_{\text{cyclic}}(a^3 - bc(b+c))\right)(a : b : c). \end{aligned}$$

This latter point is the triangle center

$$X_{57} = (\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c})$$

in [2], which divides the segment OI in the ratio $OX_{57} : OI = 2R + r : 2R - r$. The common poles F and F', it follows from (5) above, divide the segment IX_{57} harmonically in the ratio $2R - r : \pm\sqrt{(4R+r)r}$.

References

[1] N. Altshiller-Court, *College Geometry*, 2nd edition, 1952, Barnes and Noble, New York.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*,
http://cedar.evansville.edu/ ck6/encyclopedia/ .
[3] P. Yiu, Hyacinthos messages 999 and 1004, June, 2000,
http://groups.yahoo.com/group/Hyacinthos.

Edward Brisse: 5628 Saint-André, Montréal, Québec, H2S 2K1, Canada
E-mail address: ebcfgl@videotron.ca