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Simple Constructions of the Incircle of an Arbelos

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Abstract. We give several simple constructions of the incircle of an arbelos, also known as a shoemaker's knife.

Archimedes, in his *Book of Lemmas*, studied the arbelos bounded by three semicircles with diameters AB, AC, and CB, all on the same side of the diameters.[1] See Figure 1. Among other things, he determined the radius of the incircle of the arbelos. In Figure 2, GH is the diameter of the incircle parallel to the base AB, and G', H' are the (orthogonal) projections of G, H on AB. Archimedes showed that $GHH'G'$ is a square, and that AG', $G'H'$, $H'B$ are in geometric progression. See [1, pp. 307–308].

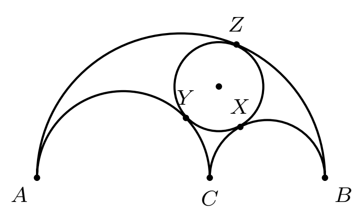


Figure 1

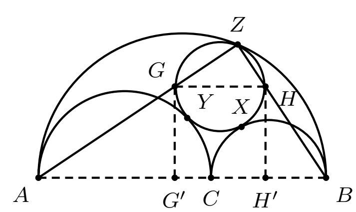
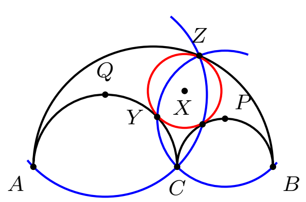
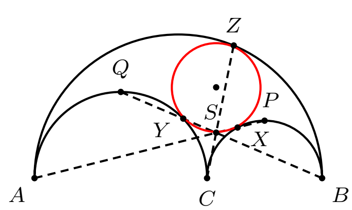


Figure 2

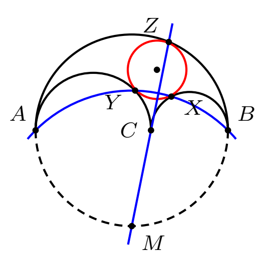
In this note we give several simple constructions of the incircle of the arbelos. The elegant Construction 1 below was given by Leon Bankoff [2]. The points of tangency are constructed by drawing circles with centers at the midpoints of two of the semicircles of the arbelos. In validating Bankoff's construction, we obtain Constructions 2 and 3, which are easier in the sense that one is a ruler-only construction, and the other makes use only of the midpoint of one semicircle.



Construction 1



Construction 2



Construction 3

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[1]The arbelos is also known as the shoemaker's knife. See [3].

Theorem 1 (Bankoff [2]). *Let P and Q be the midpoints of the semicircles (BC) and (AC) respectively. If the incircle of the arbelos is tangent to the semicircles (BC), (AC), and (AB) at X, Y, Z respectively, then*
(i) A, C, X, Z lie on a circle, center Q;
(ii) B, C, Y, Z lie on a circle, center P.

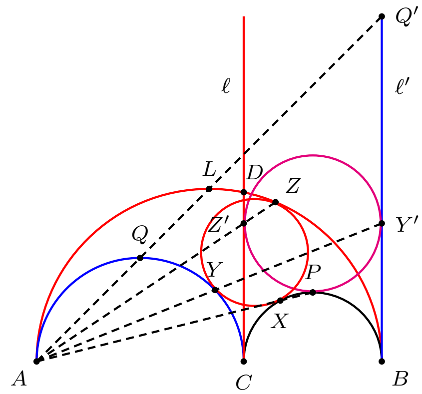


Figure 3

Proof. Let D be the intersection of the semicircle (AB) with the line perpendicular to AB at C. See Figure 3. Note that $AB \cdot AC = AD^2$ by Euclid's proof of the Pythagorean theorem. [2] Consider the inversion with respect to the circle $A(D)$. This interchanges the points B and C, and leaves the line AB invariant. The inversive images of the semicircles (AB) and (AC) are the lines ℓ and ℓ perpendicular to AB at C and B respectively. The semicircle (BC), being orthogonal to the invariant line AB, is also invariant under the inversion. The incircle XYZ of the arbelos is inverted into a circle tangent to the semicircle (BC), and the lines ℓ, ℓ, at P, Y', Z' respectively. Since the semicircle (BC) is invariant, the points A, X, and P are collinear. The points Y' and Z' are such that BPZ' and CPY' are lines making 45° angles with the line AB. Now, the line BPZ' also passes through the midpoint L of the semicircle (AB). The inversive image of this line is a circle passing through A, C, X, Z. Since inversion is conformal, this circle also makes a 45° angle with the line AB. Its center is therefore the midpoint Q of the semicircle (AC). This proves that the points X and Z lies on the circle $Q(A)$.

The same reasoning applied to the inversion in the circle $B(D)$ shows that Y and Z lie on the circle $P(B)$. □

Theorem 1 justifies Construction 1. The above proof actually gives another construction of the incircle XYZ of the arbelos. It is, first of all, easy to construct the circle $PY'Z'$. The points X, Y, Z are then the intersections of the lines AP, AY', and AZ' with the semicircles (BC), (CA), and (AB) respectively. The following two interesting corollaries justify Constructions 2 and 3.

[2]*Euclid's Elements*, Book I, Proposition 47.

Corollary 2. *The lines AX, BY, and CZ intersect at a point S on the incircle XYZ of the arbelos.*

Proof. We have already proved that A, X, P are collinear, as are B, Y, Q. In Figure 4, let S be the intersection of the line AP with the circle XYZ. The inversive image S' (in the circle $A(D)$) is the intersection of the same line with the circle $PY'Z'$. Note that

$$\angle AS'Z' = \angle PS'Z' = \angle PY'Z' = 45^\circ = \angle ABZ'$$

so that A, B, S', Z' are concyclic. Considering the inversive image of this circle, we conclude that the line CZ contains S. In other words, the lines AP and CZ intersect at the point S on the circle XYZ. Likewise, BQ and CZ intersect at the same point. □

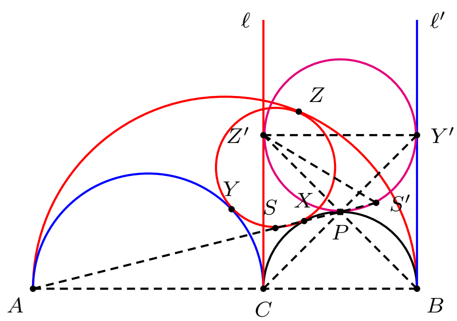


Figure 4

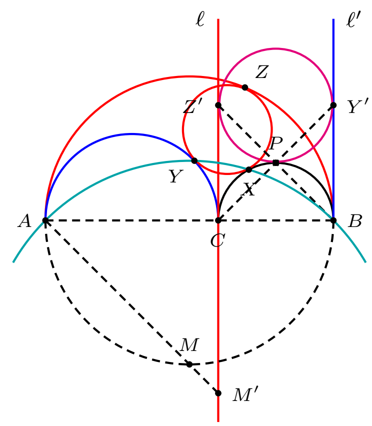


Figure 5

Corollary 3. *Let M be the midpoint of the semicircle (AB) on the opposite side of the arbelos.*

(i) The points A, B, X, Y lie on a circle, center M.

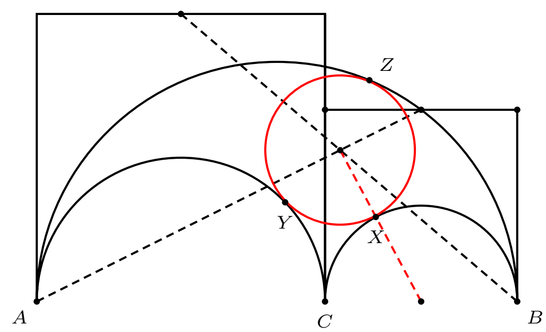
(ii) The line CZ passes through M.

Proof. Consider Figure 5 which is a modification of Figure 3. Since C, P, Y' are on a line making a 45° angle with AB, its inversive image (in the circle $A(D)$) is a circle through A, B, X, Y, also making a 45° angle with AB. The center of this circle is necessarily the midpoint M of the semicircle AB on the opposite side of the arbelos.

Join A, M to intersect the line ℓ at M'. Since $\angle BAM' = 45^\circ = \angle BZ'M'$, the four points A, Z', B, M' are concyclic. Considering the inversive image of the circle, we conclude that the line CZ passes through M. □

The center of the incircle can now be constructed as the intersection of the lines joining X, Y, Z to the centers of the corresponding semicircles of the arbelos.

However, a closer look into Figure 4 reveals a simpler way of locating the center of the incircle XYZ. The circles XYZ and $PY'Z'$, being inversive images, have the center of inversion A as a center of similitude. This means that the center of the incircle XYZ lies on the line joining A to the midpoint of $Y'Z'$, which is the opposite side of the square erected on BC, on the same side of the arbelos. The same is true for the square erected on AC. This leads to the following Construction 4 of the incircle of the arbelos:



Construction 4

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[1] T.L. Heath, *The Works of Archimedes with the Method of Archimedes*, 1912, Dover reprint; also in *Great Books of the Western World*, 11, Encyclopædia Britannica Inc., Chicago, 1952.

[2] L. Bankoff, A mere coincide, *Mathematics Newsletter*, Los Angeles City College, November 1954; reprinted in *College Math. Journal* 23 (1992) 106.

[3] C.W. Dodge, T. Schoch, P.Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, *Mathematics Magazine* 72 (1999) 202–213.

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