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The Gergonne problem

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Abstract. An effective method for the proof of geometric inequalities is the use of the dot product of vectors. In this paper we use this method to solve some famous problems, namely Heron's problem, Fermat's problem and the extension of the previous problem in space, the so called Gergonne's problem. The solution of this last is erroneously stated, but not proved, in F.G.-M.

1. Introduction

In this paper whenever we write AB we mean the length of the vector $\mathbf{AB}$, i.e. $AB = |\mathbf{AB}|$. The method of using the dot product of vectors to prove geometric inequalities consists of using the following well known properties:

(1) $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$.

(2) $\mathbf{a} \cdot \mathbf{i} \leq \mathbf{a} \cdot \mathbf{j}$ if $\mathbf{i}$ and $\mathbf{j}$ are unit vectors and $\angle(\mathbf{a}, \mathbf{i}) \geq \angle(\mathbf{a}, \mathbf{j})$.

(3) If $\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|}$ is the unit vector along $\mathbf{AB}$, then the length of the segment AB is given by

$$AB = \mathbf{i} \cdot \mathbf{AB}.$$

2. The Heron problem and the Fermat point

2.1. *Heron's problem.* A point O on a line XY gives the smallest sum of distances from the points A, B (on the same side of XY) if $\angle XOA = \angle BOY$.

Proof. If M is an arbitrary point on XY (see Figure 1) and $\mathbf{i}, \mathbf{j}$ are the unit vectors of $\mathbf{OA}, \mathbf{OB}$ respectively, then the vector $\mathbf{i} + \mathbf{j}$ is perpendicular to XY since it bisects the angle between $\mathbf{i}$ and $\mathbf{j}$. Hence $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{OM} = 0$ and

$$\begin{aligned} OA + OB &= \mathbf{i} \cdot \mathbf{OA} + \mathbf{j} \cdot \mathbf{OB} \\ &= \mathbf{i} \cdot (\mathbf{OM} + \mathbf{MA}) + \mathbf{j} \cdot (\mathbf{OM} + \mathbf{MB}) \\ &= (\mathbf{i} + \mathbf{j}) \cdot \mathbf{OM} + \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} \\ &= \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} \\ &\leq |\mathbf{i}||\mathbf{MA}| + |\mathbf{j}||\mathbf{MB}| \\ &= MA + MB. \end{aligned}$$

□

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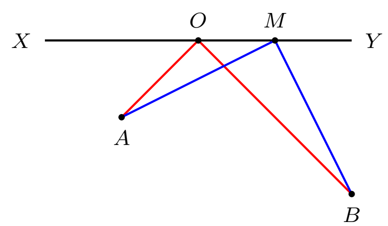

Figure 1

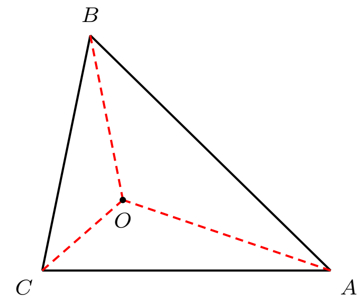

Figure 2

2.2. *The Fermat point.* If none of the angles of a triangle ABC exceeds $120°$, the point O inside a triangle ABC such that $\angle BOC = \angle COA = \angle AOB = 120°$ gives the smallest sum of distances from the vertices of ABC. See Figure 2.

Proof. If M is an arbitrary point and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors of $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$, then $\mathbf{i}+\mathbf{j}+\mathbf{k}=\mathbf{0}$ since this vector does not changes by a $120°$ rotation. Hence,

$$\begin{aligned}
OA+OB+OC &= \mathbf{i}\cdot\mathbf{OA}+\mathbf{j}\cdot\mathbf{OB}+\mathbf{k}\cdot\mathbf{OC}\\
&= \mathbf{i}\cdot(\mathbf{OM}+\mathbf{MA})+\mathbf{j}\cdot(\mathbf{OM}+\mathbf{MB})+\mathbf{k}\cdot(\mathbf{OM}+\mathbf{MC})\\
&= (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}+\mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\
&= \mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\
&\leq |\mathbf{i}||\mathbf{MA}|+|\mathbf{j}||\mathbf{MB}|+|\mathbf{k}||\mathbf{MC}|\\
&= MA+MB+MC.
\end{aligned}$$

□

3. The Gergonne problem

Given a plane π and a triangle ABC not lying in the plane, the Gergonne problem [3] asks for a point O on a plane π such that the sum $OA+OB+OC$ is minimum. This is an extention of Fermat's problem to 3 dimensions. According to [2, pp. 927–928], [1] this problem had hitherto been unsolved (for at least 90 years). Unfortunately, as we show in §4.1 below, the solution given there, for the special case when the planes π and ABC are parallel, is erroneous. We present a solution here in terms of the centroidal line of a trihedron. We recall the definition which is based on the following fact. See, for example, [1, p.43].

Proposition and Definition. *The three planes determined by the edges of a trihedral angle and the internal bisectors of the respective opposite faces intersect in a line. This line is called the centroidal line of the trihedron.*

Theorem 1. *If O is a point on the plane π such that the centroidal line of the trihedron $O.ABC$ is perpendicular to π, then $OA+OB+OC \leq MA+MB+MC$ for every point M on π.*

[1] Problem 742-III, especially 1901 c3.

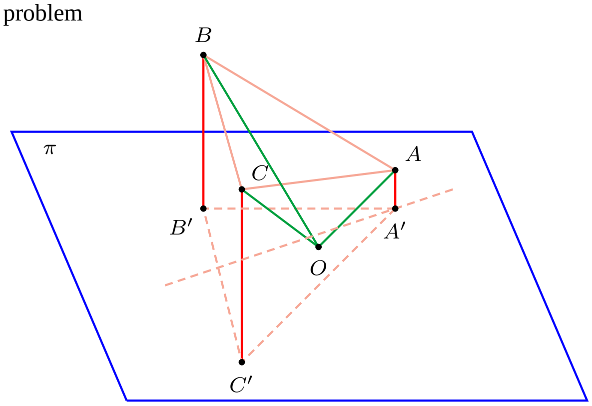

Figure 3

Proof. Let M be an arbitrary point on π, and **i**, **j**, **k** the unit vectors along $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$ respectively. The vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is parallel to the centroidal line of the trihedron $O.ABC$. Since this line is perpendicular to π by hypothesis we have

$$(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}=0. \tag{1}$$

Hence,

$$\begin{aligned} OA+OB+OC &= \mathbf{i}\cdot\mathbf{OA}+\mathbf{j}\cdot\mathbf{OB}+\mathbf{k}\cdot\mathbf{OC} \\ &= \mathbf{i}\cdot(\mathbf{OM}+\mathbf{MA})+\mathbf{j}\cdot(\mathbf{OM}+\mathbf{MB})+\mathbf{k}\cdot(\mathbf{OM}+\mathbf{MC}) \\ &= (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}+\mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC} \\ &= \mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC} \\ &\le |\mathbf{i}||\mathbf{MA}|+|\mathbf{j}||\mathbf{MB}|+|\mathbf{k}||\mathbf{MC}| \\ &= MA+MB+MC. \end{aligned}$$

□

4. Examples

We set up a rectangular coordinate system such that A, B, C, are the points $(a,0,p)$, $(0,b,q)$ and $(0,c,r)$. Let A', B', C' be the orthogonal projections of A, B, C on the plane π. Write the coordinates of O as $(x,y,0)$. The x- and y-axes are the altitude from A' and the line $B'C'$ of triangle $A'B'C'$ in the plane π. Since

$$\begin{aligned} \mathbf{i} &= \frac{-1}{\sqrt{(x-a)^2+y^2+p^2}}(x-a,y,-p), \\ \mathbf{j} &= \frac{-1}{\sqrt{x^2+(y-b)^2+q^2}}(x,y-b,-q), \\ \mathbf{k} &= \frac{-1}{\sqrt{x^2+(y-c)^2+r^2}}(x,y-c,-r), \end{aligned}$$

it is sufficient to put in (1) for $\mathbf{OM}$ the vectors $(1,0,0)$ and $(0,1,0)$. From these, we have

$$\begin{aligned}\frac{x-a}{\sqrt{(x-a)^2+y^2+p^2}}+\frac{x}{\sqrt{x^2+(y-b)^2+q^2}}+\frac{x}{\sqrt{x^2+(y-c)^2+r^2}} &= 0,\\ \frac{y}{\sqrt{(x-a)^2+y^2+p^2}}+\frac{y-b}{\sqrt{x^2+(y-b)^2+q^2}}+\frac{y-c}{\sqrt{x^2+(y-c)^2+r^2}} &= 0.\end{aligned} \tag{2}$$

The solution of this system cannot in general be expressed in terms of radicals, as it leads to equations of high degree. It is therefore in general not possible to construct the point O using straight edge and compass. We present several examples in which O is constructible. In each of these examples, the underlying geometry dictates that $y = 0$, and the corresponding equation can be easily written down.

4.1. *π parallel to ABC.* It is very easy to mistake for O the Fermat point of triangle $A'B'C'$, as in [2, loc. cit.]. If we take $p = q = r = 3$, $a = 14$, $b = 2$, and $c = -2$, the system (2) gives $y = 0$ and

$$\frac{x-14}{\sqrt{(x-14)^2+9}}+\frac{2x}{\sqrt{x^2+13}}=0, \qquad x>0.$$

This leads to the quartic equation

$$3x^4-84x^3+611x^2+364x-2548=0.$$

This quartic polynomial factors as $(x-2)(3x^3-78x^2+455x+1274)$, and the only positive root of which is $x = 2$. [2] Hence $\angle B'OC' = 90°$, $\angle A'OB' = 135°$, and $\angle A'OC' = 135°$, showing that O is not the Fermat point of triangle $A'B'C'$. [3]

4.2. *ABC isosceles with A on π and BC parallel to π.* In this case, $p = 0$, $q = r$, $c = -b$, and we may assume $a > 0$. The system (2) reduces to $y = 0$ and

$$\frac{x-a}{|x-a|}+\frac{2x}{\sqrt{x^2+b^2+q^2}}=0.$$

Since $0 < x < a$, we get

$$(x,y)=\left(\sqrt{\frac{b^2+q^2}{3}},\ 0\right)$$

with $b^2 + q^2 < 3a^2$. Geometrically, since $OB = OC$, the vectors $\mathbf{i}$, $\mathbf{j}-\mathbf{k}$ are parallel to π. We have

$$\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=0, \qquad (\mathbf{i}+\mathbf{j}+\mathbf{k})(\mathbf{j}-\mathbf{k})=0.$$

Equivalently,

$$\mathbf{i}\cdot\mathbf{j}+\mathbf{i}\cdot\mathbf{k}=-1, \qquad \mathbf{i}\cdot\mathbf{j}-\mathbf{i}\cdot\mathbf{k}=0.$$

Thus, $\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = -\frac{1}{2}$ or $\angle AOB = \angle AOC = 120°$, a fact that is a generalization of the Fermat point to 3 dimensions.

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[2] The cubic factor has one negative root ≈ -2.03472, and two non-real roots. If, on the other hand, we take $p = q = r = 2$, the resulting equation becomes $3x^4-84x^3+596x^2+224x-1568 = 0$, which is irreducible over rational numbers. It roots are not constructible using ruler and compass. The positive real root is $x \approx 1.60536$. There is a negative root ≈ -1.61542 and two non-real roots.

[3] The solution given in [2] assumes erroneously OA, OB, OC equally inclined to the planes π and of triangle ABC.

If $b^2+q^2 \geq 3a^2$, the centroidal line cannot be perpendicular to π, and Theorem 1 does not help. In this case we take as point O to be the intersection of x-axis and the plane MBC. It is obvious that

$$MA+MB+MC \geq OA+OB+OC = |x-a| + 2\sqrt{x^2+b^2+q^2}.$$

We write $f(x) = |x-a| + 2\sqrt{x^2+b^2+q^2}$.

If $0 < a < x$, then $f'(x) = 1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} > 0$ and f is an increasing function. For $x \leq 0$, $f'(x) = -1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} < 0$ and f is a decreasing function.

If $0 < x \leq a \leq \sqrt{\frac{b^2+q^2}{3}}$, then $4x^2 \leq x^2+b^2+q^2$ so that $f'(x) = -1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} \leq 0$ and f is a decreasing function. Hence we have minimum when $x = a$ and $O \equiv A$.

4.3. *B, C on π.* If the points B, C lie on π, then the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is perpendicular to the vectors $\mathbf{j}$ and $\mathbf{k}$. From these, we obtain the interesting equality $\angle AOB = \angle AOC$. Note that they are not neccesarily equal to 120°, as in Fermat's case. Here is an example. If $a = 10$, $b = 8$, $c = -8$, $p = 3$, $q = r = 0$ the system (2) gives $y = 0$ and

$$\frac{x-10}{\sqrt{(x-10)^2+9}} + \frac{2x}{\sqrt{x^2+64}} = 0, \qquad 0 < x < 10,$$

which leads to the equation

$$3x^4 - 60x^3 + 272x^2 + 1280x - 6400 = 0.$$

This quartic polynomial factors as $(x-4)(3x^3-48x^2+80x+1600)$. It follows that the only positive root is $x = 4$. [4] Hence we have

$$\angle AOB = \angle AOC = \arccos(-\frac{2}{5}) \qquad \text{and} \qquad \angle BOC = \arccos(-\frac{3}{5}).$$

References

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, 2nd ed., Chelsea reprint, 1964.
[2] F.G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, 1991, Paris.
[3] J.-D. Gergonne, *Annales mathématiques de Gergonne*, 12 (1821-1822) 380.

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[4] The cubic factor has one negative root ≈ -4.49225, and two non-real roots.