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Equilateral Triangles Intercepted by Oriented Parallelians

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Abstract. Given a point P in the plane of triangle ABC, we consider rays through P parallel to the side lines. The intercepts on the sidelines form an equilateral triangle precisely when P is a Brocardian point of one of the Fermat points. There are exactly four such equilateral triangles.

1. Introduction

The construction of an interesting geometric figure is best carried out after an analysis. For example, given a triangle ABC, how does one construct a point P through which the parallels to the three sides make equal intercepts? A very simple analysis of this question can be found in [6, 7]. It is shown that there is only one such point P,[1] which has homogeneous barycentric coordinates

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}:\frac{1}{c}+\frac{1}{a}-\frac{1}{b}:\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\sim(ca+ab-bc:ab+bc-ca:bc+ca-ab).$$

This leads to a very easy construction of the point[2] and its three equal parallel intercepts. See Figure 1. An interesting variation is to consider equal "semi-parallel

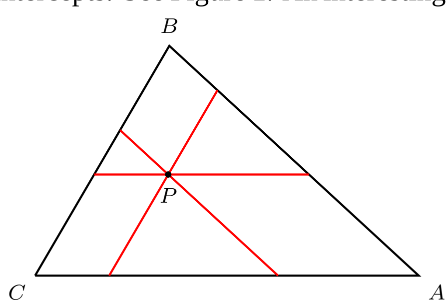

Figure 1

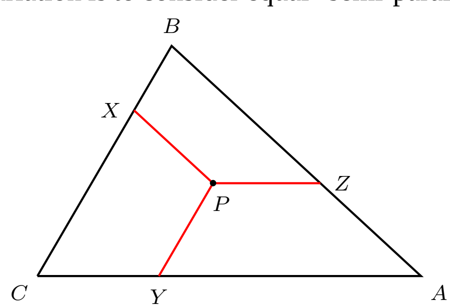

Figure 2

intercepts". Suppose through a point P in the plane of triangle ABC, parallels to the sides AB, BC, CA intersect BC, CA, AB are X, Y, Z respectively. How should one choose P so that the three "semi-parallel intercepts" PX, PY, PZ have equal lengths? (Figure 2). A simple calculation shows that the only point satisfying this requirement, which we denote by $L_{\rightarrow}$, has coordinates $(\frac{1}{c}:\frac{1}{a}:\frac{1}{b})$.

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[1] In [3], this is the equal-parallelian point X_{192}. In [7], this is called the equal-intercept point.

[2] If G is the centroid and I' the isotomic conjugate of the incenter of triangle ABC, then $I'P = 3\cdot I'G$.

If we reverse the orientations of the parallel rays, we obtain another point $J_{\leftarrow}$ with coordinates $(\frac{1}{b} : \frac{1}{c} : \frac{1}{a})$. See Figure 5. These two points are called the Jerabek points; they can be found in [2, p.1213]. For a construction, see §4.

2. Triangles intercepted by forward parallelians

Given a triangle ABC, we mean by a *parallelian* a directed ray parallel to one of the sides, *forward* if it is along the direction of AB, BC, or CA, and *backward* if it is along BA, CB, or AC. In this paper we study the question: how should one choose the point P so that so that the *triangle XYZ intercepted by forward parallelians through P* is equilateral? See Figure 2. [3] We solve this problem by performing an analysis using homogeneous barycentric coordinates. If $P = (u : v : w)$, then X, Y, and Z have coordinates

$$X = (0 : u + v : w), \quad Y = (u : 0 : v + w), \quad Z = (w + u : v : 0).$$

The lengths of AY and AZ are respectively $\frac{(v+w)b}{u+v+w}$ and $\frac{vc}{u+v+w}$. By the law of cosines, the square length of YZ is

$$\frac{1}{(u+v+w)^2}((v+w)^2b^2 + v^2c^2 - (v+w)v(b^2 + c^2 - a^2)).$$

Similarly, the square lengths of ZX and XY are respectively

$$\frac{1}{(u+v+w)^2}((w+u)^2c^2 + w^2a^2 - (w+u)w(c^2 + a^2 - b^2))$$

and

$$\frac{1}{(u+v+w)^2}((u+v)^2a^2 + u^2b^2 - (u+v)u(a^2 + b^2 - c^2)).$$

The triangle XYZ is equilateral if and only if

$$\begin{aligned} & (v+w)^2b^2 + v^2c^2 - (v+w)v(b^2 + c^2 - a^2) \\ = \; & (w+u)^2c^2 + w^2a^2 - (w+u)w(c^2 + a^2 - b^2) \\ = \; & (u+v)^2a^2 + u^2b^2 - (u+v)u(a^2 + b^2 - c^2). \end{aligned} \tag{1}$$

By taking differences of these expressions, we rewrite (1) as a system of two homogeneous quadratic equations in three unknowns:

$$\mathcal{C}_1 : \qquad a^2v^2 - b^2w^2 - ((b^2 + c^2 - a^2)w - (c^2 + a^2 - b^2)v)u = 0,$$

and

$$\mathcal{C}_2 : \qquad b^2w^2 - c^2u^2 - ((c^2 + a^2 - b^2)u - (a^2 + b^2 - c^2)w)v = 0.$$

[3] Clearly, a solution to this problem can be easily adapted to the case of “backward triangles”, as we shall do at the end §4.

3. Intersections of two conics

3.1. *Representation by symmetric matrices.* We regard each of the two equations $\mathcal{C}_1$ and $\mathcal{C}_2$ as defining a conic in the plane of triangle ABC. The question is therefore finding the intersections of two conics. This is done by choosing a suitable combination of the two conics which degenerates into a pair of straight lines. To do this, we represent the two conics by symmetric 3×3 matrices

$$M_1 = \begin{pmatrix} 0 & c^2+a^2-b^2 & -(b^2+c^2-a^2) \\ c^2+a^2-b^2 & 2a^2 & 0 \\ -(b^2+c^2-a^2) & 0 & -2b^2 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} -2c^2 & -(c^2+a^2-b^2) & 0 \\ -(c^2+a^2-b^2) & 0 & a^2+b^2-c^2 \\ 0 & a^2+b^2-c^2 & 2b^2 \end{pmatrix},$$

and choose a combination $M_1 - tM_2$ whose determinant is zero.

3.2. *Reduction to the intersection with a pair of lines.* Consider, therefore, the matrix

$$M_1 - tM_2 = \begin{pmatrix} 2tc^2 & (1+t)(c^2+a^2-b^2) & -(b^2+c^2-a^2) \\ (1+t)(c^2+a^2-b^2) & 2a^2 & -t(a^2+b^2-c^2) \\ -(b^2+c^2-a^2) & -t(a^2+b^2-c^2) & -2(1+t)b^2 \end{pmatrix}. \tag{2}$$

Direct calculation shows that the matrix $M_1 - tM_2$ in (2) has determinant

$$-32\triangle^2((b^2-c^2)t^3 - (c^2+a^2-2b^2)t^2 - (c^2+a^2-2b^2)t - (a^2-b^2)),$$

where $\triangle$ denotes the area of triangle ABC. The polynomial factor further splits into

$$((b^2-c^2)t - (a^2-b^2))(t^2+t+1).$$

We obtain $M_1 - tM_2$ of determinant zero by choosing $t = \dfrac{a^2-b^2}{b^2-c^2}$. This matrix represents a quadratic form which splits into two linear forms. In fact, the combination $(b^2-c^2)\mathcal{C}_1 - (a^2-b^2)\mathcal{C}_2$ leads to

$$((a^2-b^2)u + (b^2-c^2)v + (c^2-a^2)w)(c^2u + a^2v + b^2w) = 0.$$

From this we see that the intersections of the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same as those of any one of them with the pairs of lines

ℓ_1 : $$(a^2-b^2)u + (b^2-c^2)v + (c^2-a^2)w = 0,$$

and

ℓ_2 : $$c^2u + a^2v + b^2w = 0.$$

3.3. *Intersections of* $\mathcal{C}_1$ *with* ℓ_1 *and* ℓ_2. There is an easy parametrization of points on the line ℓ_1. Since it clearly contains the points $(1:1:1)$ (the centroid) and $(c^2 : a^2 : b^2)$, every point on ℓ_1 is of the form $(c^2+t : a^2+t : b^2+t)$ for some real number t. Direct substitution shows that this point lies on the conic $\mathcal{C}_1$ if and only if

$$3t^2 + 3(a^2+b^2+c^2)t + (a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2) = 0.$$

In other words,

$$\begin{aligned} t &= \frac{-(a^2+b^2+c^2)}{2} \pm \frac{1}{2\sqrt{3}}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4} \\ &= \frac{-(a^2+b^2+c^2)}{2} \pm \frac{2\triangle}{\sqrt{3}}. \end{aligned}$$

From these, we conclude that the conic $\mathcal{C}_1$ and the line ℓ_1 intersect at the points

$$P^{\pm} = \left(\frac{a^2+b^2-c^2}{2} \pm \frac{2\triangle}{\sqrt{3}} : \frac{b^2+c^2-a^2}{2} \pm \frac{2\triangle}{\sqrt{3}} : \frac{c^2+a^2-b^2}{2} \pm \frac{2\triangle}{\sqrt{3}}\right). \quad (3)$$

The line ℓ_2, on the other hand, does not intersect the conic $\mathcal{C}_1$ at real points. [4] It follows that the conics $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect only at the two real points $P^{\pm}$ given in (3) above. [5]

4. Construction of the points $P^{\pm}$

The coordinates of $P^{\pm}$ in (3) can be rewritten as

$$\begin{aligned} P^{\pm} &= (ab\cos C \pm \frac{1}{\sqrt{3}}ab\sin C : bc\cos A \pm \frac{1}{\sqrt{3}}bc\sin A : ca\cos B \pm \frac{1}{\sqrt{3}}ca\sin B) \\ &= (\frac{2ab}{\sqrt{3}}\sin(C \pm \frac{\pi}{3}) : \frac{2bc}{\sqrt{3}}\sin(A \pm \frac{\pi}{3}) : \frac{2ca}{\sqrt{3}}\sin(B \pm \frac{\pi}{3})) \\ &\sim (\frac{1}{c}\cdot\sin(C \pm \frac{\pi}{3}) : \frac{1}{a}\cdot\sin(A \pm \frac{\pi}{3}) : \frac{1}{b}\cdot\sin(B \pm \frac{\pi}{3})). \end{aligned}$$

A simple interpretation of these expressions, via the notion of Brocardian points [5], leads to an easy construction of the points $P^{\pm}$.

Definition. The Brocardian points of a point $Q = (x : y : z)$ are the two points

$$Q_{\rightarrow} = (\frac{1}{z} : \frac{1}{x} : \frac{1}{y}) \quad \text{and} \quad Q_{\leftarrow} = (\frac{1}{y} : \frac{1}{z} : \frac{1}{x}).$$

We distinguish between these two by calling $Q_{\rightarrow}$ the *forward* Brocardian point and $Q_{\leftarrow}$ the *backward* one, and justify these definitions by giving a simple construction.

Proposition 1. *Given a point Q, construct through the traces A_Q, B_Q, C_Q forward parallelians to AB, BC, CA, intersecting CA, AB, BC at Y, Z and X respectively. The lines AX, BY, CZ intersect at $Q_{\rightarrow}$. On the other hand, if the*

[4] Substitution of $u = \frac{-(a^2v+b^2w)}{c^2}$ into $(\mathcal{C}_1)$ gives $a^2v^2+(a^2+b^2-c^2)vw+b^2w^2 = 0$, which has no real roots since $(a^2+b^2-c^2)^2-4a^2b^2 = a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2 = -16\triangle^2 < 0$.

[5] See Figure 9 in the Appendix for an illustration of the conics and their intersections.

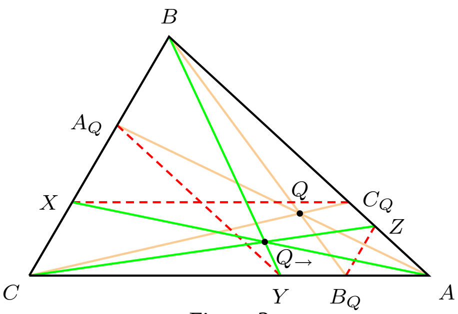

Figure 3a

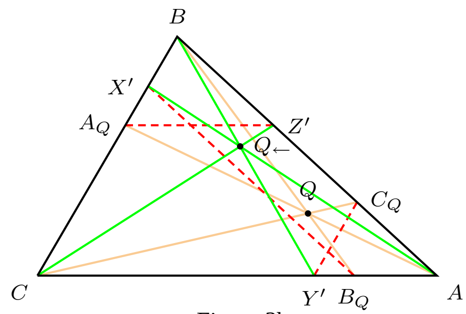

Figure 3b

backward parallelians through A_Q, B_Q, C_Q to CA, AB, BC, intersect AB, BC, CA at Z', X', Y' respectively, then, the lines AX', BY', CZ' intersect at $Q_\leftarrow$.

Proof. Suppose $Q = (x : y : z)$ in homogeneous barycentric coordinates. In Figure 3a, $BX : XC = BC_Q : C_QA = x : y$ since $C_Q = (x : y : 0)$. It follows that $X = (0 : y : x) \sim (0 : \frac{1}{x} : \frac{1}{y})$. Similarly, $Y = (\frac{1}{z} : 0 : \frac{1}{x})$ and $Z = (\frac{1}{z} : \frac{1}{y} : 0)$. From these, the lines AX, BY, and CZ intersect at the point $(\frac{1}{z} : \frac{1}{x} : \frac{1}{y})$, which we denote by $Q_\rightarrow$. The proof for $Q_\leftarrow$ is similar; see Figure 3b. $\square$

Examples. If $Q = K = (a^2 : b^2 : c^2)$, the symmedian point, the Brocardian points $K_\rightarrow$ and $K_\leftarrow$ are the Brocard points [6] satisfying

$$\angle K_\rightarrow BA = \angle K_\rightarrow CB = \angle K_\rightarrow AC = \omega = \angle K_\leftarrow CA = \angle K_\leftarrow AB = \angle K_\leftarrow BC,$$

where ω is the Brocard angle given by $\cot\omega = \cot A + \cot B + \cot C$. These points lie on the circle with OK as diameter, O being the circumcenter of triangle ABC. See Figure 4.

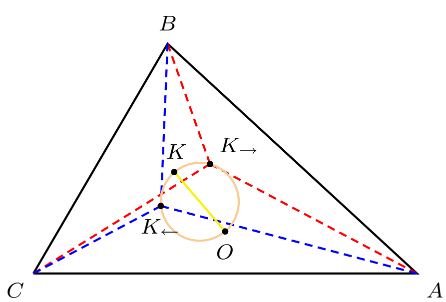

Figure 4

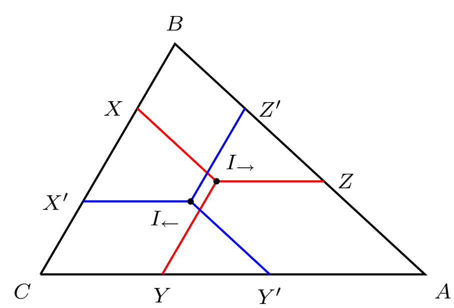

Figure 5

On the other hand, the Brocardian points of the incenter $I = (a : b : c)$ are the Jerabek points $I_\rightarrow$ and $I_\leftarrow$ mentioned in §1. See Figure 5.

[6] These points are traditionally labelled Ω (for $K_\rightarrow$) and Ω' (for $K_\leftarrow$) respectively. See [1, pp.274–280.]

Proposition 2. *The points $P^{\pm}$ are the* forward *Brocardian points of the Fermat points* [7]

$$F^{\pm} = \left(\frac{a}{\sin(A \pm \frac{\pi}{3})} : \frac{b}{\sin(B \pm \frac{\pi}{3})} : \frac{c}{\sin(C \pm \frac{\pi}{3})} \right).$$

By reversing the orientation of the parallelians, we obtain two more equilateral triangles, corresponding to the *backward* Brocardian points of the same two Fermat points $F^{\pm}$.

Theorem 3. *There are exactly four equilateral triangles intercepted by oriented parallelians, corresponding to the four points $F^{\pm}_{\rightarrow}$ and $F^{\pm}_{\leftarrow}$.*

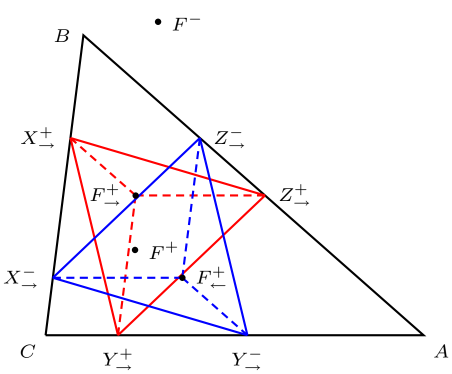

Figure 6a

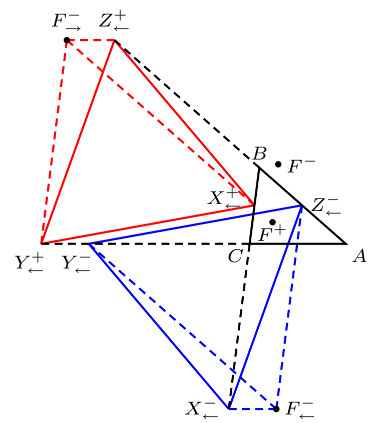

Figure 6b

5. Some further results

The two equilateral triangles $X^+_{\rightarrow}Y^+_{\rightarrow}Z^+_{\rightarrow}$ and $X^+_{\leftarrow}Y^+_{\leftarrow}Z^+_{\leftarrow}$ corresponding to the Fermat point F^+ are congruent; so are $X^-_{\rightarrow}Y^-_{\rightarrow}Z^-_{\rightarrow}$ and $X^-_{\leftarrow}Y^-_{\leftarrow}Z^-_{\leftarrow}$. In fact, they are homothetic at the common midpoint of the segments $X^+_{\rightarrow}Y^+_{\leftarrow}$, $Y^+_{\rightarrow}Z^+_{\leftarrow}$, and $Z^+_{\rightarrow}X^+_{\leftarrow}$, and their sides are parallel to the corresponding cevians of the Fermat point. This is indeed a special case of the following proposition.

Proposition 4. *For every point Q not on the side lines of triangle ABC, the triangle intercepted by the forward parallelians through $Q_{\rightarrow}$ and that by the backward parallelians through $Q_{\leftarrow}$ are homothetic at $(u(v+w) : v(w+u) : w(u+v))$, with ratio $1 : -1$. Their corresponding sides are parallel to the cevians AQ, BQ, and CQ respectively.*

[7]The Fermat point F^+ (respectively F^-) of triangle ABC is the intersection of the lines AX, BY, CZ, where XBC, YCA and ZAB are equilateral triangles constructed externally (respectively internally) on the sides BC, CA, AB of the triangle. This is the point X_{13} (respectively X_{14}) in [3].

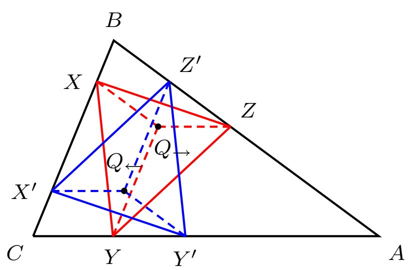

Figure 7

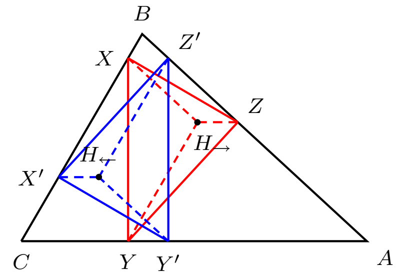

Figure 8

These two triangles are the only inscribed triangles whose sides are parallel to the respective cevians of Q. See Figure 7. They are the Bottema triangles in [4]. Applying this to the orthocenter H, we obtain the two congruent inscribed triangles whose sides are perpendicular to the sides of ABC (Figure 8).

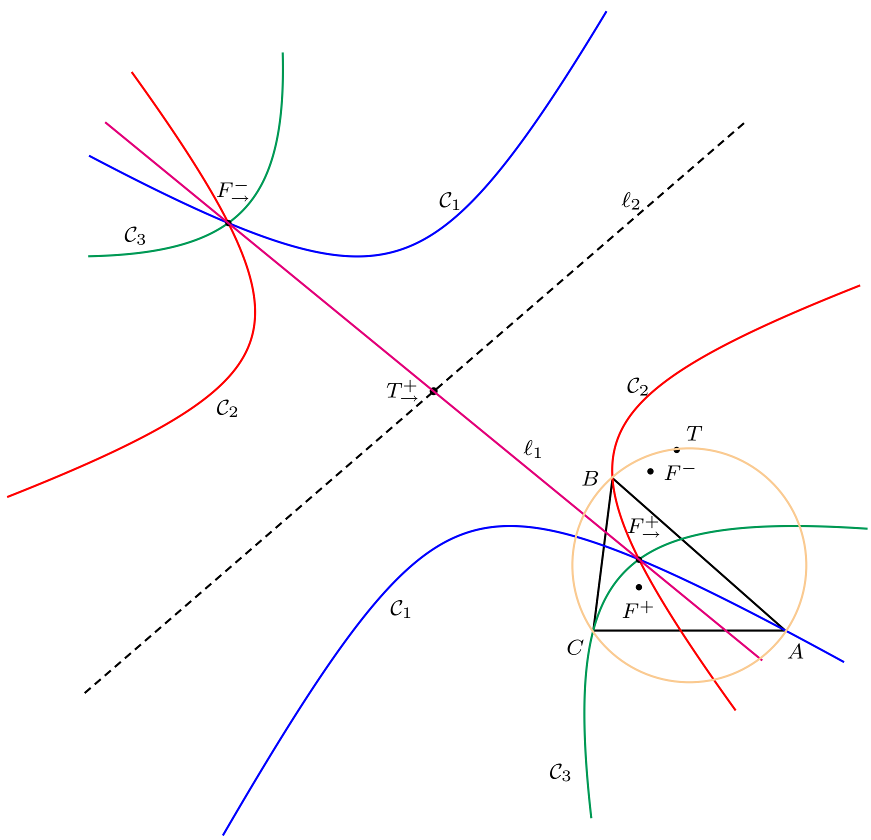

Figure 9

Appendix

Figure 9 illustrates the intersections of the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ in §2, along with a third conic $\mathcal{C}_3$ which results from the difference of the first two expressions in (1), namely,

$$\mathcal{C}_3: \qquad c^2u^2 - a^2v^2 - ((a^2+b^2-c^2)v - (b^2+c^2-a^2)u)w = 0.$$

These three conics are all hyperbolas, and have a common center $T^+_{\rightarrow}$, which is the forward Brocardian point of the Tarry point T, and is the midpoint between the common points $F^+_{\rightarrow}$ and $F^-_{\rightarrow}$. In other words, $F^+_{\rightarrow}F^-_{\rightarrow}$ is a common diameter of the three hyperbolas. We remark that the Tarry point T is the point X_{98} of [3], and is the fourth intersection of the Kiepert hyperbola and the circumcircle of triangle ABC. The fact that ℓ_1 and ℓ_2 intersect at $T_{\rightarrow}$ follows from the observation that these lines are respectively the loci of the forward Brocardians of points on the Kiepert hyperbola $\frac{b^2-c^2}{u} + \frac{c^2-a^2}{v} + \frac{a^2-b^2}{w} = 0$ and the circumcircle $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ respectively. The tangents to the hyperbolas $\mathcal{C}_1$ at A, $\mathcal{C}_2$ at B, and $\mathcal{C}_3$ at C intersect at the point $H_{\rightarrow}$, the forward Brocardian of the orthocenter.

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