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Congruent Inscribed Rectangles

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Abstract. We solve the construction problem of an interior point P in a given triangle ABC with congruent rectangles inscribed in the subtriangles PBC, PCA and PAB.

1. Congruent inscribed rectangles

Given a triangle with sidelengths a, b, c, let $L_m = \min(a, b, c)$; $L \in (0, L_m)$ and $\mu > 0$. Let P be a point inside ABC with distances d_a, d_b, d_c to the sidelines of ABC. Suppose that a rectangle with lengths of sides L and μL is inscribed in the triangle PBC, with two vertices with distance L on the segment BC, the other vertices on the segments PB and PC. Then, $\dfrac{L}{d_a - \mu L} = \dfrac{a}{d_a}$, or $d_a = \dfrac{\mu a L}{a - L}$.

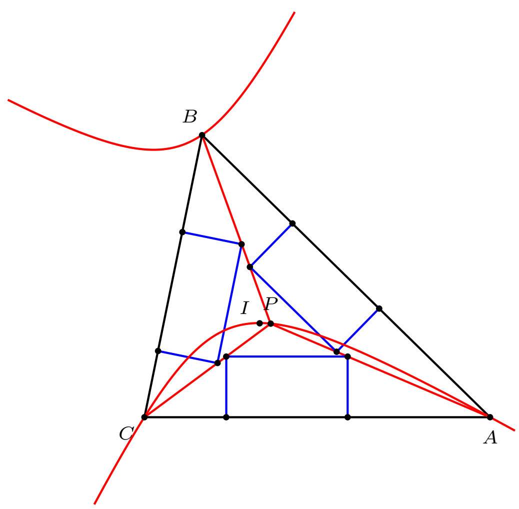

Figure 1

If we can inscribe congruent rectangles with side lengths L and μL in the three triangles PBC, PCA, PAB, we have necessarily

$$f_\mu(L) := \frac{a^2}{a - L} + \frac{b^2}{b - L} + \frac{c^2}{c - L} - \frac{2\Delta}{\mu L} = 0, \qquad (1)$$

where Δ is the area of triangle ABC. This is because $ad_a + bd_b + cd_c = 2\Delta$.

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The function $f_\mu(L)$ increases from $-\infty$ to $+\infty$ when L moves on $(0, L_m)$. The equation $f_\mu(L) = 0$ has a unique root L_μ in $(0, L_m)$ and the point

$$P_\mu = \left(\frac{a^2}{a - L_\mu} : \frac{b^2}{b - L_\mu} : \frac{c^2}{c - L_\mu}\right)$$

in homogeneous barycentric coordinates is the only point P inside ABC for which we can inscribe congruent rectangles with side lengths L_μ and μL_μ in the three triangles PBC, PCA, PAB. If $\mathcal{H}_0$ is the circumhyperbola through I (incenter) and K (symmedian point), the locus of P_μ when μ moves on $(0, +\infty)$ is the open arc Ω of $\mathcal{H}_0$ from I to the vertex of ABC opposite to the shortest side. See Figure 1. For $\mu = 1$, the smallest root L_1 of $f_1(L) = 0$ leads to the point P_1 with congruent inscribed squares.

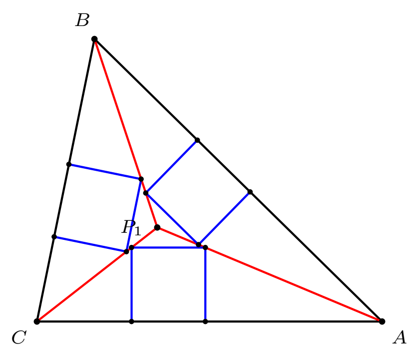

Figure 2

2. Construction of congruent inscribed rectangles

Consider $P \in \Omega$, Q and E the reflections of P and C with respect to the line IB. The parallel to AB through Q intersects BP at F. The lines EF and AP intersect at X. Then the parallel to AB through X is a sideline of the rectangle inscribed in PAB. The reflections of this line with respect to AI and BI will each give a sideline of the two other rectangles. [1]

Proof. We have $\dfrac{\overline{BE}}{\overline{BA}} = \dfrac{a}{c}$, $\dfrac{\overline{BP}}{\overline{BF}} = \dfrac{d_c}{d_a} = \dfrac{c}{a}\dfrac{a - L_\mu}{c - L_\mu}$. Applying the Menelaus theorem to triangle PAB and transversal EFX, we have

$$\frac{\overline{XA}}{\overline{XP}} = \frac{\overline{FB}}{\overline{FP}}\frac{\overline{EA}}{\overline{EB}} = \frac{L_\mu - c}{L_\mu}.$$

More over, the sidelines of the rectangles parallel to BC, CA, AB form a triangle homothetic at I with ABC. □

[1]This construction was given by Bernard Gibert.

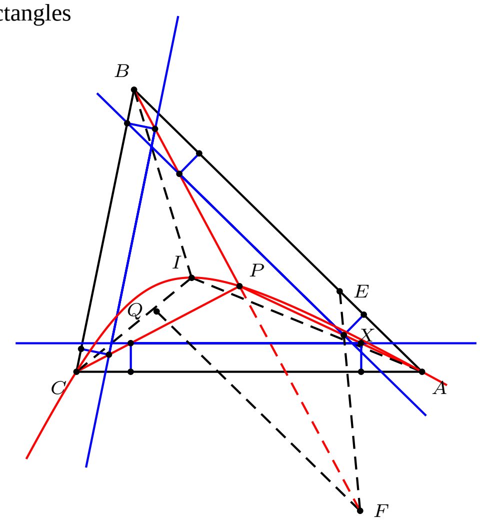

Figure 3

3. Construction of P_μ

The point P_μ is in general not constructible with ruler and compass. We give here a construction as the intersection of the arc Ω with a circle.

Consider the points

$$X_{100} = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$$

and

$$X_{106} = \left(\frac{a^2}{b+c-2a} : \frac{b^2}{c+a-2b} : \frac{c^2}{a+b-2c}\right)$$

on the circumcircle. [2] Note that the line $X_{100}X_{106}$ passes through the incenter I. The line joining X_{106} to the symmedian point K intersects the circumcircle again at

$$X_{101} = \left(\frac{a^2}{b-c} : \frac{b^2}{c-a} : \frac{c^2}{a-b}\right).$$

Construction. Draw outwardly a line ℓ parallel to AC at a distance μb from AC, intersecting the line CK at S. The parallel at S to the line CX_{101} intersects the line $KX_{101}X_{106}$ at Y_μ. Then P_μ is the intersection of the arc Ω with the circle through X_{100}, X_{106}, and Y_μ. See Figure 4.

Proof. From

$$L = \frac{2a\Delta x}{2\Delta x + \mu a^2(x+y+z)} = \frac{2b\Delta y}{2\Delta x + \mu b^2(x+y+z)} = \frac{2c\Delta z}{2\Delta z + \mu a^2(x+y+z)},$$

[2]We follow the notations of [1]. Here, X_{100} is the isogonal conjugate of the infinite point of the trilinear polar of the incenter, and X_{106} is the isogonal conjugate of the infinite point of the line GI joining the centroid and the incenter.

we note that P_μ lies on the three hyperbolas $\mathcal{H}_a$, $\mathcal{H}_b$ and $\mathcal{H}_c$ with equations

$$\mu bc(x+y+z)(cy-bz)+2\Delta(b-c)yz = 0, \qquad (\mathcal{H}_a)$$
$$\mu ca(x+y+z)(az-cx)+2\Delta(c-a)zx = 0, \qquad (\mathcal{H}_b)$$
$$\mu ab(x+y+z)(bx-ay)+2\Delta(a-b)xy = 0. \qquad (\mathcal{H}_c)$$

Computing $a^2(a-b)(c-a)(\mathcal{H}_a)+b^2(b-c)(a-b)(\mathcal{H}_b)+c^2(c-a)(b-c)(\mathcal{H}_c)$, we see that P_μ lies on the circle Γ_μ:

$$\mu abc(x+y+z)\Lambda+2\Delta(a-b)(b-c)(c-a)(a^2yz+b^2zx+c^2xy)=0,$$

where

$$\Lambda = bc(b-c)(b+c-2a)x+ca(c-a)(c+a-2b)y+ab(a+b-2c)(a-b)z.$$

As $\Lambda=0$ is the line $X_{100}X_{106}$, the circle Γ_μ passes through X_{100} and X_{106}.

Now, as ℓ is the line $2\Delta y+\mu b^2(x+y+z)=0$, we have

$$S=\left(a^2 : b^2 : -\left(a^2+b^2+\frac{2\Delta}{\mu}\right)\right).$$

The parallel through S to CX_{101} is the line

$$\mu(b+a-2c)(x+y+z)+2\Delta\left(\frac{(b-c)x}{a^2}+\frac{(a-c)y}{b^2}\right)=0,$$

and KX_{101} is the line

$$b^2c^2(b-c)(b+c-2a)x+c^2a^2(c-a)(c+a-2b)y+a^2b^2(a-b)(a+b-2c)z=0.$$

We can check that these two lines intersect at the point

$$\begin{aligned}Y_\mu =&(a^2(2\Delta(c-a)(a-b)+\mu(-a^2(b^2+c^2)+2abc(b+c)+(b^4-2b^3c-2bc^3+c^4))\\ &:b^2(2\Delta(a-b)(b-c)+\mu(-b^2(c^2+a^2)+2abc(c+a)+(c^4-2c^4a-2ca^3+a^4))\\ &:c^2(2\Delta(b-c)(c-a)+\mu(-c^2(a^2+b^2)+2abc(a+b)+(a^4-2a^3b-2ab^3+b^4)))\end{aligned}$$

on the circle Γ_μ. □

Remark. The circle through X_{100}, X_{106} and P_μ is the only constructible circle through P_μ, and there is no constructible line through P_μ.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/~ck6/encyclopedia/.

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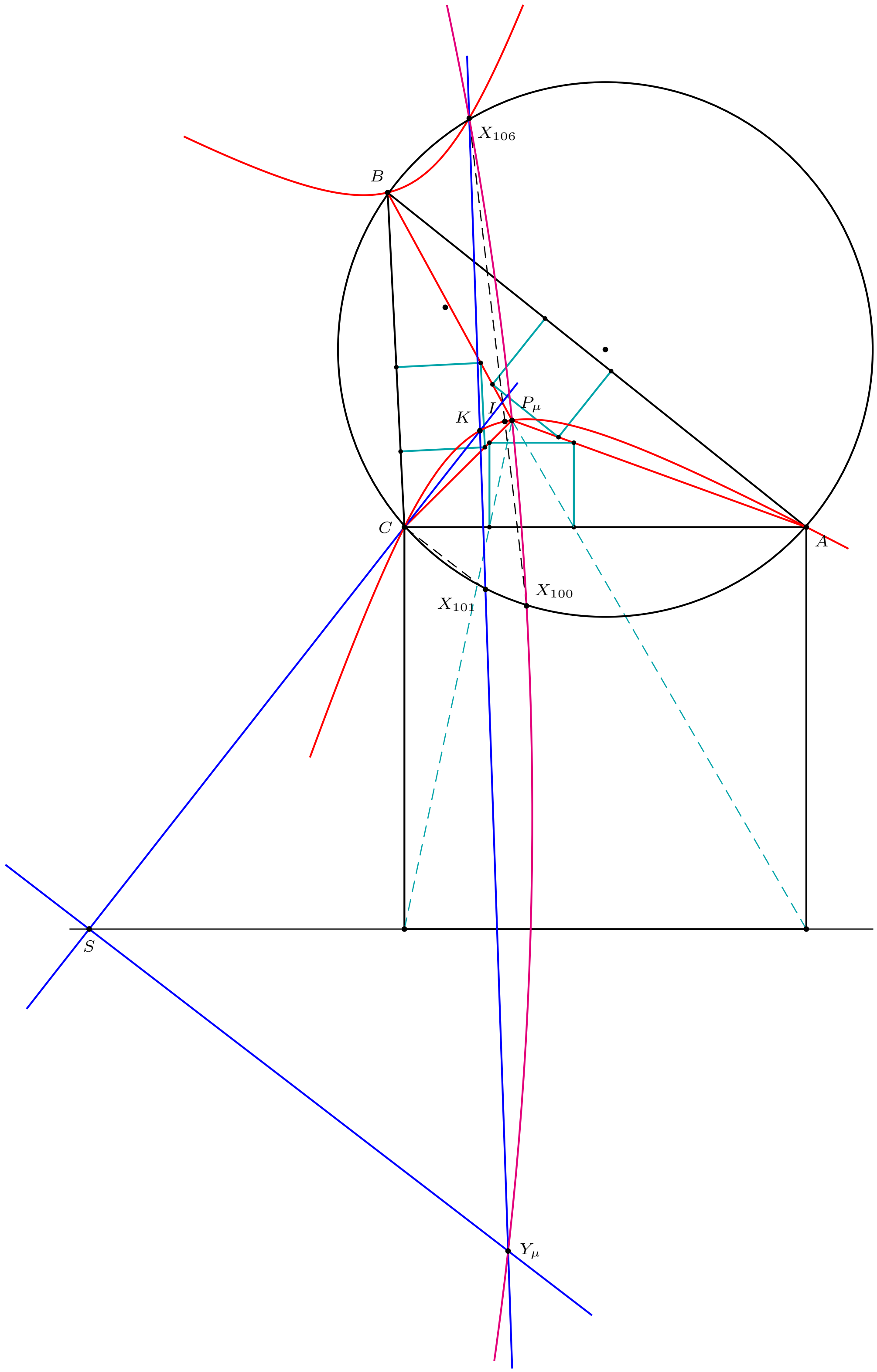

Figure 4