Forum Geometricorum
Volume 2 (2002) 67–70.

FORUM GEOM
ISSN 1534-1178

A Rapid Construction of Some Triangle Centers

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Abstract. We give a compass and ruler construction of fifteen centers associated with a triangle by drawing 6 circles and 23 lines.

Given triangle T with vertices A, B, and C, draw a red circle centered at A passing through B, another centered at B going through C, and a third centered at C going through A. Now, draw a blue circle centered at A passing through C, one centered at C going through B, and one centered at B going through A. There will be 12 intersections of red circles with blue ones. Three of them are A, B, and C. Three are apices of equilateral triangles erected on the sides of T and pointing outward. Denote such an apex by A_+, B_+, C_+. Three are the apices of equilateral triangles erected on the sides pointing inward. Denote them by A_-, B_-, C_-. The last three are the reflections of the vertices of T in the opposite sides, which we shall call A^*, B^*, C^*.

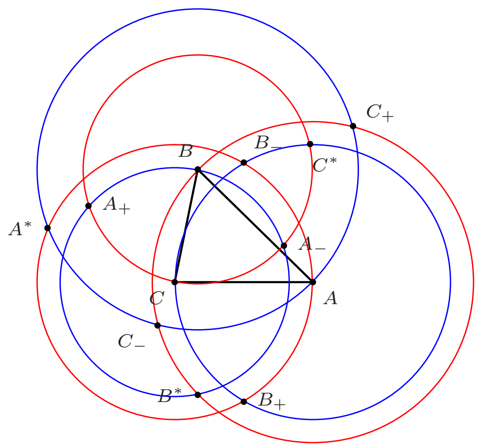

Figure 1. Construction of $A_\pm$, $B_\pm$, $C_\pm$, A^*, B^*, C^*

The four triangles $T = ABC$, $T_+ = A_+B_+C_+$, $T_- = A_-B_-C_-$, and $T^* = A^*B^*C^*$ are pairwise in perspective. The 6 centers of perspectivity are

(1) $[T, T_+] = F_+$, the inner Fermat point,
(2) $[T, T_-] = F_-$, the outer Fermat point,
(3) $[T, T^*] = H$, the orthocenter,

Publication Date: June 18, 2002. Communicating Editor: Paul Yiu.
Expanded from a presentation at the MAA Meeting, Urbana, Illinois, March 22, 2001.

(4) $[T_+, T_-] = O$, the circumcenter,
(5) $[T_+, T^*] = J_-$, the outer isodynamic point,
(6) $[T_-, T^*] = J_+$, the inner isodynamic point.

Only two lines, AA_+ and BB_+, are needed to determine F_+ by intersection. Likewise, 10 more are necessary to determine the other 5 centers F_-, H, O, J_- and J_+. We have drawn twelve lines so far. [1] See Figure 2, where the green lines only serve to indicate perspectivity; they are not necessary for the constructions of the triangle centers.

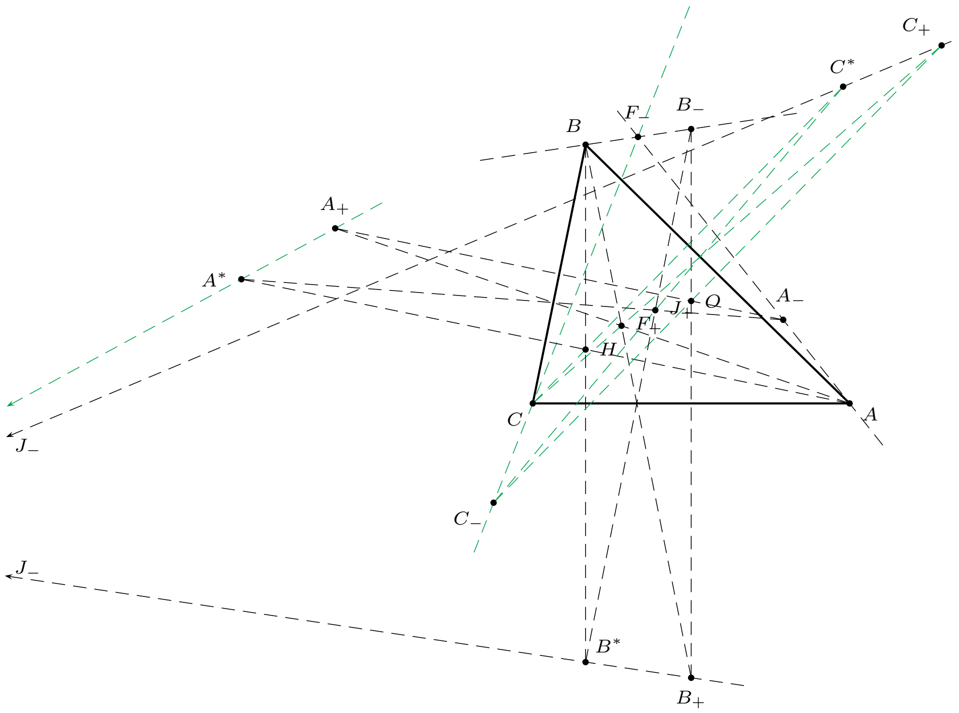

Figure 2. Construction of $F_\pm$, H, O, $J_\pm$

Define three more lines: the Euler line OH, the Fermat line F_+F_-, and the Apollonius line J_+J_-. The Apollonius line J_+J_- is also known as the Brocard axis. It contains the circumcenter O and the (Lemoine) symmedian point K. Then,

(7) $K = J_+J_- \cap F_+F_-$;
(8) $D = OH \cap F_+F_-$ is the center of orthocentroidal circle, the midpoint of between the centroid and the orthocenter.

We construct six more lines to locate four more centers:

(9) the outer Napoleon point is $N_+ = HJ_+ \cap OF_+$,

[1] The 18 points A, $A_\pm$, A^*, B, $B_\pm$, B^*, C, $C_\pm$, C^*, H, O, $F\pm$, $J_\pm$ all lie on a third degree curve called the Neuberg cubic.

(10) the inner Napoleon point is $N_- = HJ_- \cap OF_-$;
(11) the centroid $G = OH \cap J_+F_-$ (or $OH \cap J_-F_+$);
(12) the nine-point center $N_p = OH \cap N_-F_+$ (or $OH \cap N_+F-$).

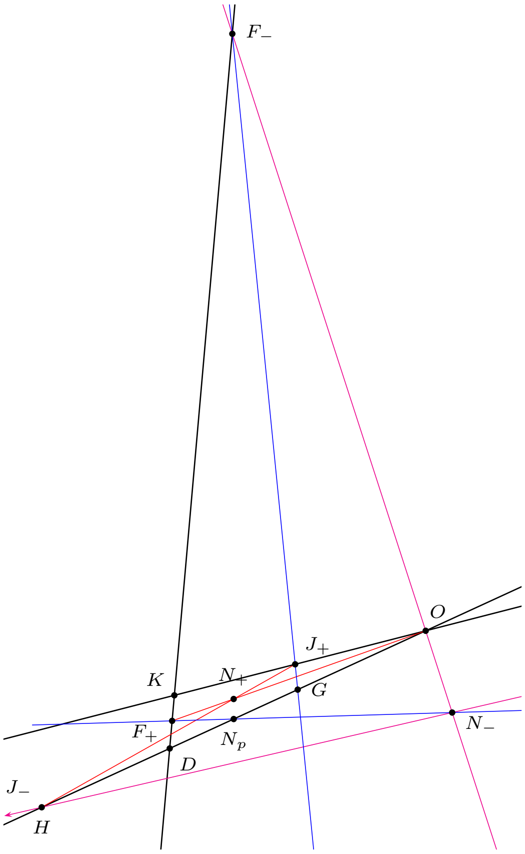

Figure 3. Construction of K, D, $N_\pm$, G, N_p

The line N_-F_+ (used in (12) above to locate N_p) intersects $OK = J_+J_-$ at the isogonal conjugate of N_-. Likewise, the lines N_+F_- and OK intersect at the isogonal conjugate of N_+. We also note that the line J_+N_- intersects the Euler line OH at the nine-point center N_p' of the medial triangle. Thus,

(13) $N_+^* = N_+F_- \cap OK$,
(14) $N_-^* = N_-F_+ \cap OK$, and
(15) $N_p' = J_+N_- \cap OH$ (or $J_-N_+ \cap OH$).

See Figure 4, in which we note that the points G, N_+ and N_-^* are collinear, so are G, N_- and N_+^*.

We have therefore constructed 15 centers with 6 circles and 23 lines: 12 to determine O, H, $F_\pm$, $J_\pm$ as the 6 centers of perspectivity of T, $T_\pm$ and T^*; then 9 to determine K, D, $N_\pm$, G, N_p, N_-^*, and finally 2 more to give N_+^* and N_p'.

Remark. The triangle centers in this note appear in [1, 2] as X_n for n given below.

center	O	H	F_+	F_-	J_+	J_-	K	D	N_+	N_-	G	N_p	N_+^*	N_-^*	N_p'
n	3	4	13	14	15	16	6	381	17	18	2	5	61	62	140

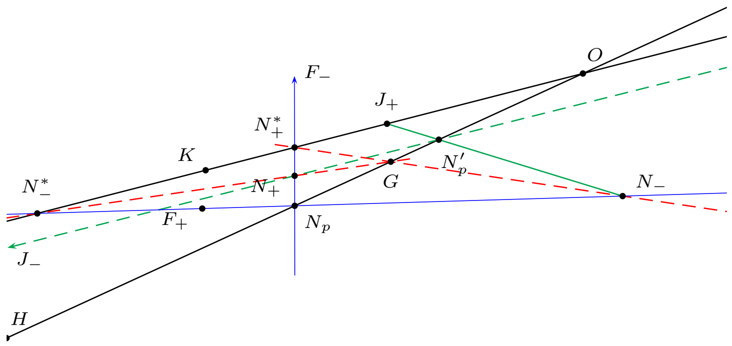

Figure 4. Construction of N_+^*, N_-^*, and N_p'

This construction uses Kimberling's list [1] of collinearities among centers. It can be implemented on a dynamic software like the Geometer's Sketchpad. After hiding the circles and lines, one is left with T and the centers, which can be observed to move in concert as one drags a vertex of T on the computer screen. Some important centers we do not get here are the incenter, the Gergonne and the Nagel points.

References

[1] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 295.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.

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