Forum Geometricorum
Volume 2 (2002) 71–87.

FORUM GEOM
ISSN 1534-1178

A Generalization of the Tucker Circles

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Abstract. Let hexagon $PQRSTU$ be inscribed in triangle $A_1A_2A_3$ (ordered counterclockwise) such that P and S are on line A_3A_1, Q and T are on line A_1A_2, and R and U are on line A_2A_3. If PQ, RS, and TU are respectively parallel to A_2A_3, A_1A_2, and A_3A_1, while QR, ST, and UP are antiparallel to A_3A_1, A_2A_3, and A_1A_2 respectively, the vertices of the hexagon are on one circle. Now, let hexagon $P'Q'R'S'T'U'$ be described as above, with each of its sides parallel to the corresponding side of $PQRSTU$. Again the six vertices are concyclic, and the process may be repeated indefinitely to form an infinite family of circles (Tucker [3]). This family is a coaxaloid system, and its locus of centers is the Brocard axis of the triangle, passing through the circumcenter and the symmedian point. J. A. Third ([2]) extended this idea by relaxing the conditions for the directions of the sides of the hexagon, thus finding infinitely many coaxaloid systems of circles. The present paper defines a further extension by allowing the directions of the sides to be as arbitrary as possible, resulting in families of homothetic conics with properties analogous to those of the Tucker circles.

1. Circles of Tucker and Third

The system of Tucker circles is a special case of the systems of Third circles. In a Third system the directions of PQ, QR, and RS may be taken arbitrarily, while ST is made antiparallel to PQ (with respect to angle $A_2A_1A_3$). Similarly, TU and UP are made antiparallel to QR and RS respectively. The hexagon may then be inscribed in a circle, and a different starting point P' with the same directions produces another circle. It should be noted that the six vertices need not be confined to the sides of the triangle; each point may lie anywhere on its respective sideline. Thus an infinite family of circles may be obtained, and Third shows that this is a coaxaloid system. That is, it may be derived from a coaxal system of circles by multiplying every radius by a constant. (See Figures 1a and 1b). In particular, the Tucker system is obtained from the coaxal system of circles through the Brocard points Ω and Ω' by multiplying the radius of each circle by $\dfrac{R}{O\Omega}$, R being the circumradius of the triangle and O its circumcenter ([1, p.276]). In general, the line of centers of a Third system is the perpendicular bisector of the segment joining the pair of isogonal conjugate points which are the common points of the corresponding coaxal system. Furthermore, although the coaxal system has no envelope, it

Publication Date: July 5, 2002. Communicating Editor: Clark Kimberling.

will be seen later that the envelope of the coaxaloid system is a conic tangent to the sidelines of the triangle, whose foci are the points common to the coaxal circles.

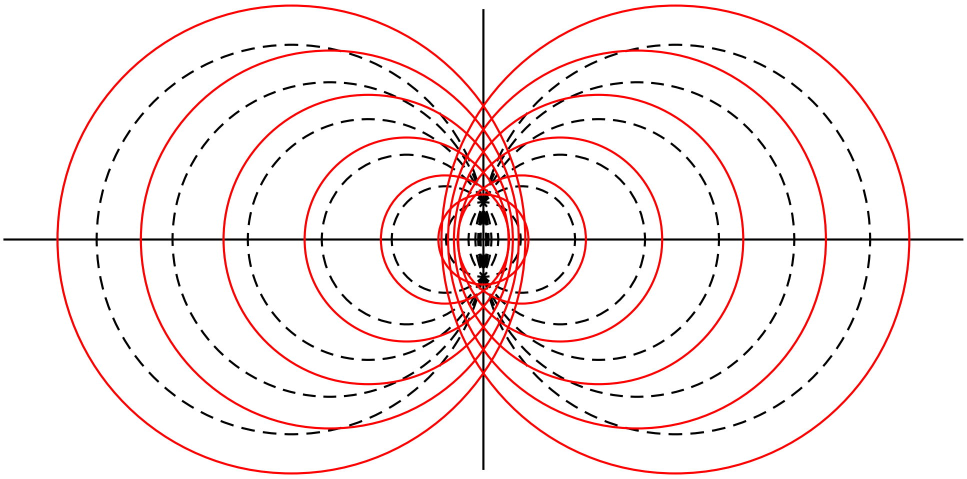

Figure 1a: Coaxaloid system with elliptic envelope, and its corresponding coaxal system

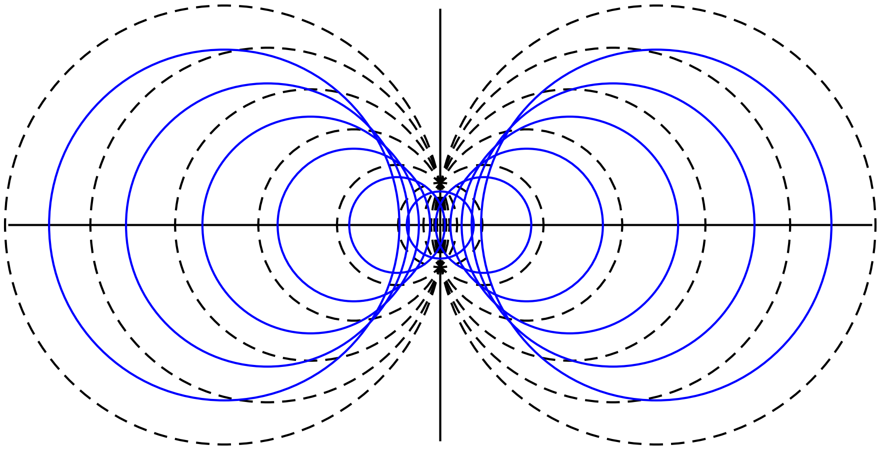

Figure 1b: Coaxaloid system with hyperbolic envelope, and its corresponding coaxal system

2. Two-circuit closed paths in a triangle

2.1. Consider a polygonal path from P on A_3A_1 to Q on A_1A_2 to R on A_2A_3 to S on A_3A_1 to T on A_1A_2 to U on A_2A_3, and back to P. Again the six points may be selected anywhere on their respective sidelines. The vertices of the triangle are numbered counterclockwise, and the lengths of the corresponding sides are denoted by a_1, a_2, a_3. Distances measured along the perimeter of the triangle in the counterclockwise sense are regarded as positive. The length of PA_1 is designated by λ, which is negative in case A_1 is between A_3 and P. Thus, $A_3P = a_2 - \lambda$, and the barycentric coordinates of P are $(a_2 - \lambda : 0 : \lambda)$. Also, six "directions" w_i are defined:

$$w_1 = \frac{PA_1}{A_1Q}, \qquad w_2 = \frac{QA_2}{A_2R}, \qquad w_3 = \frac{RA_3}{A_3S},$$
$$w_4 = \frac{SA_1}{A_1T}, \qquad w_5 = \frac{TA_2}{A_2U}, \qquad w_6 = \frac{UA_3}{A_3P}.$$

Any direction may be positive or negative depending on the signs of the directed segments. Then, $A_1Q = \dfrac{\lambda}{w_1}$, $QA_2 = \dfrac{a_3w_1 - \lambda}{w_1}$, $A_2R = \dfrac{a_3w_1 - \lambda}{w_1w_2}$, and so on.

2.2. A familiar example is that in which PQ and ST are parallel to A_2A_3, QR and TU are parallel to A_3A_1, and RS and UP are parallel to A_1A_2 (Figure 2). Then

$$w_1 = w_4 = \frac{a_2}{a_3}, \qquad w_2 = w_5 = \frac{a_3}{a_1}, \qquad w_3 = w_6 = \frac{a_1}{a_2}.$$

It is easily seen by elementary geometry that this path closes after two circuits around the sidelines of the triangle.

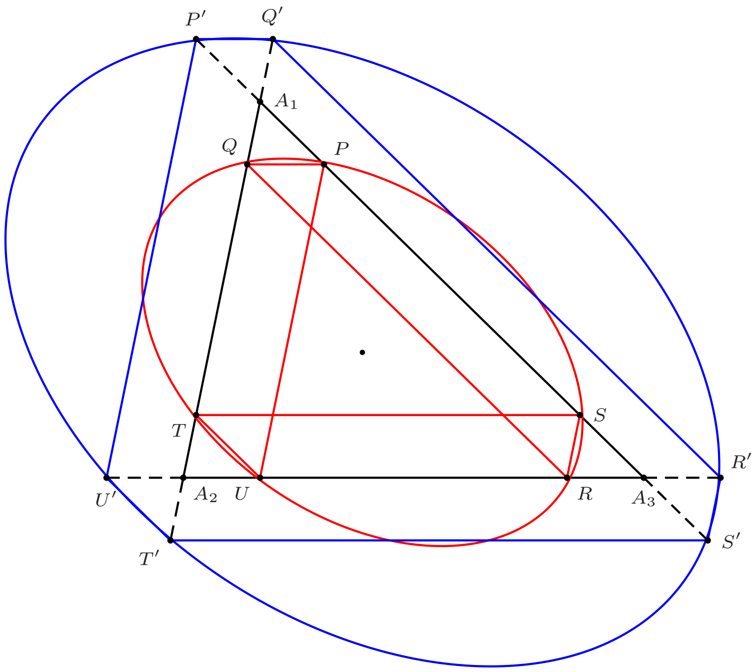

Figure 2. Hexagonal paths formed by parallels

2.3. Closure is less obvious, but still not difficult to prove, when "parallel" in the first example is replaced by "antiparallel" (Figure 3). Here,

$$w_1 = w_4 = \frac{a_3}{a_2}, \qquad w_2 = w_5 = \frac{a_1}{a_3}, \qquad w_3 = w_6 = \frac{a_2}{a_1}.$$

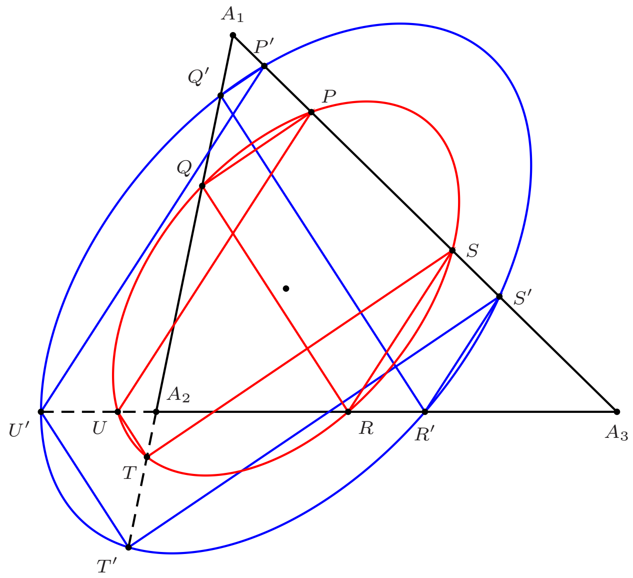

Figure 3. Hexagonal paths formed by antiparallels

2.4. Another positive result is obtained by using isoscelizers ([1, p.93]). That is, $PA_1 = A_1Q$, $QA_2 = A_2R$, $RA_3 = A_3S, \ldots, UA_3 = A_3P$. Therefore,

$$w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 1.$$

2.5. These examples suggest that, if $w_1 = w_4$, $w_2 = w_5$, $w_3 = w_6$, the condition $w_1w_2w_3 = 1$ is sufficient to close the path after two circuits. Indeed, by computing lengths of segments around the triangle, one obtains

$$A_3P = \frac{UA_3}{w_3} = \frac{a_1w_1^2w_2^2w_3 - a_3w_1^2w_2w_3 + a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda}{w_1^2w_2^2w_3^2}.$$

But also $A_3P = a_2 - \lambda$, and equating the two expressions yields

$$(1 - w_1w_2w_3)(a_1w_1w_2 - a_2w_1w_2w_3 - a_3w_1 + \lambda(1 + w_1w_2w_3)) = 0. \qquad (1)$$

In order that (1) may be satisfied for all values of λ, the solution is $w_1w_2w_3 = 1$.

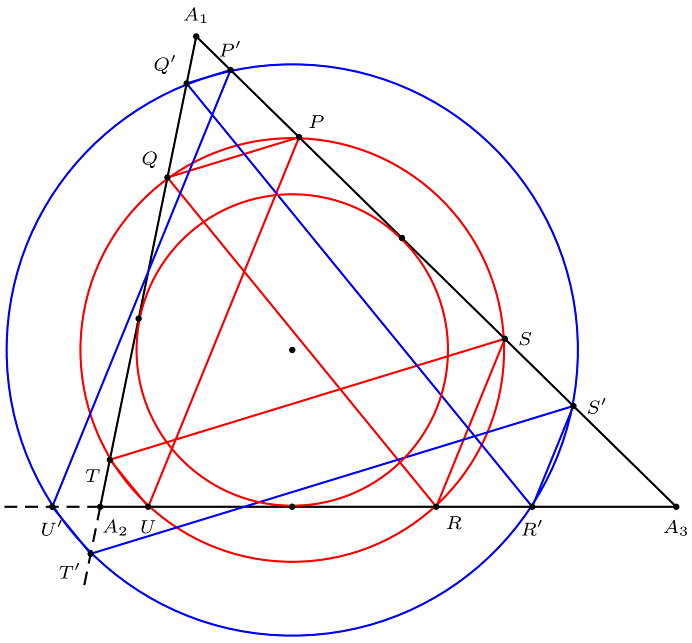

Figure 4. Hexagonal paths formed by isoscelizers

2.6. As a slight digression, the other factor in (1) gives the special solution

$$\lambda = \frac{w_1(a_2w_2w_3 - a_1w_2 + a_3)}{1 + w_1w_2w_3},$$

and calculation shows that this value of λ causes the path to close after only one circuit, that is $S = P$. For example, if antiparallels are used, and if P is the foot of the altitude from A_2, the one-circuit closed path is the orthic triangle of $A_1A_2A_3$.

Furthermore, if also $w_1w_2w_3 = 1$, the special value of λ becomes

$$\frac{a_2 - a_1w_1w_2 + a_3w_1}{2},$$

and the cevians A_1R, A_2P, and A_3Q are concurrent at the point (in barycentric coordinates, as throughout this paper)

$$\left(\frac{1}{-a_1w_1w_2 + a_2 + a_3w_1} : \frac{1}{a_1w_1w_2 - a_2 + a_3w_1} : \frac{1}{a_1w_1w_2 + a_2 - a_3w_1}\right). \tag{2}$$

It follows that there exists a conic tangent to the sidelines of the triangle at P, Q, R. The coordinates of the center of the conic are $(a_1w_1w_2 : a_2 : a_3w_1)$.

2.7. Returning to the conditions $w_1w_2w_3 = 1$, $w_1 = w_4$, $w_2 = w_5$, $w_3 = w_6$, the coordinates of the six points may be found:

$$
\begin{aligned}
P =&(a_2 - \lambda : 0 : \lambda),\\
Q =&(a_3w_1 - \lambda : \lambda : 0),\\
R =&(0 : a_1w_1w_2 - a_3w_1 + \lambda : a_3w_1 - \lambda),\\
S =&(a_1w_1w_2 - a_3w_1 + \lambda : 0 : a_2 - a_1w_1w_2 + a_3w_1 - \lambda),\\
T =&(a_1w_1w_2 - a_2 + \lambda : a_2 - a_1w_1w_2 + a_3w_1 - \lambda : 0),\\
U =&(0 : a_2 - \lambda : a_1w_1w_2 - a_2 + \lambda).
\end{aligned}
$$

These points are on one conic, given by the equation

$$
\begin{aligned}
&\lambda(a_2 - a_1w_1w_2 + a_2w_1 - \lambda)x_1^2\\
&+(a_3w_1 - \lambda)(a_1w_1w_2 - a_2 + \lambda)x_2^2\\
&+(a_2 - \lambda)(a_1w_1w_2 - a_3w_1 + \lambda)x_3^2\\
&-(a_1^2w_1^2w_2^2 + 2a_2a_3w_1 - a_3a_1w_1^2w_2 - a_1a_2w_1w_2\\
&\qquad + 2(a_1w_1w_2 - a_2 - a_3w_1)\lambda + 2\lambda^2)x_2x_3\\
&-(a_2^2 + a_2a_3w_1 - a_1a_2w_1w_2 + 2(a_1w_1w_2 - a_2 - a_3w_1)\lambda + 2\lambda^2)x_3x_1\\
&-(a_3^2w_1^2 + a_2a_3w_1 - a_3a_1w_1^2w_2 + 2(a_1w_1w_2 - a_2 - a_3w_1)\lambda + 2\lambda^2)x_1x_2\\
&=0.
\end{aligned}
\tag{3}
$$

This equation may also be written in the form

$$
\begin{aligned}
&\lambda(a_2 - a_1w_1w_2 + a_3w_1 - \lambda)(x_1 + x_2 + x_3)^2\\
&+a_3w_1(a_1w_1w_2 - a_2)x_2^2 + a_2w_1(a_1w_2 - a_3)x_3^2\\
&-(a_1^2w_1^2w_2^2 + 2a_2a_3w_1 - a_3a_1w_1^2w_2 - a_1a_2w_1w_2)x_2x_3\\
&-a_2(a_2 - a_1w_1w_2 + a_3w_1)x_3x_1\\
&-a_3w_1(a_2 - a_1w_1w_2 + a_3w_1)x_1x_2\\
&=0.
\end{aligned}
\tag{4}
$$

As λ varies, (3) or (4) represents an infinite family of conics. However, λ appears only when multiplied by $(x_1 + x_2 + x_3)^2$, so it has no effect at infinity, where $x_1 + x_2 + x_3 = 0$. Hence all conics in the system are concurrent at infinity. If they have two real points there, they are hyperbolas with respectively parallel asymptotes. This is not sufficient to make them all homothetic to each other, but it will be shown later that this is indeed the case. If the two points at infinity coincide, all of the conics are tangent to the line at infinity at that point. Therefore they are parabolas with parallel axes, forming a homethetic set. Finally, if the points at infinity are imaginary, the conics are ellipses and their asymptotes are imaginary. As in the hyperbolic case, any two conics have respectively parallel asymptotes and are homothetic to each other.

2.8. The center of (3) may be calculated by the method of [1, p.234], bearing in mind the fact that the author uses trilinear coordinates instead of barycentric. But it is easily shown that the addition of any multiple of $(x_1+x_2+x_3)^2$ to the equation of a conic has no effect on its center. Therefore (4) shows that the expression containing λ may be ignored, leaving all conics with the same center. Moreover, this center has already been found, because the conic tangent to the sidelines at $P = S$, $Q = T$, $R = U$ is a special member of (3), obtained when λ has the value $\frac{1}{2}(a_2 - a_1w_1w_2 + a_3w_1)$. Thus, the common center of all the members of (3) is $(a_1w_1w_2 : a_2 : a_3w_1)$; and if they are homothetic, any one of them may be obtained from another by a dilatation about this point. (See Figures 2, 3, 4).

Since the locus of centers is not a line, this system differs from those of Tucker and Third and may be regarded as degenerate in the context of the general thery. One case worthy of mention is that in which the sides of the hexagon are isocelizers, so that

$$w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 1.$$

Exceptionally this is a Third system, because every isocelizer is both parallel and antiparallel to itself. Therefore, the conics are concentric circles, the smallest real one being the incircle of the triangle (Figure 4).

Since the "center" of a parabola is at infinity, (3) consists of parabolas only when $a_1w_1w_2 + a_2 + a_3w_1 = 0$. This can happen if some of the directions are negative, which was seen earlier as a possibility.

2.9. Some perspectivities will now be mentioned. If

$$PU \cap QR = B_1, \quad ST \cap PU = B_2, \quad QR \cap ST = B_3,$$

the three lines A_iB_i are concurrent at (2) for every value of λ. Likewise, if

$$RS \cap TU = C_1, \quad PQ \cap RS = C_2, \quad TU \cap PQ = C_3,$$

the lines A_iC_i also concur at (2). Thus for each i the points B_i andC_i move on a fixed line through A_i.

2.10. Before consideration of the general case it may be noted that whenever the directions w_1, w_2, w_3 lead to a conic circumscribing hexagon $PQRSTU$ (that is, $w_1w_2w_3 = 1$), any permutation of them will do the same. Any permutation of w_1^{-1}, w_2^{-1}, w_3^{-1} will also work. Other such triples may be invented, such as $\frac{w_2}{w_3}$, $\frac{w_3}{w_1}$, $\frac{w_1}{w_2}$.

3. The general case

3.1. Using all six directions w_i, one may derive the following expressions for the lengths of segments:

$$
\begin{aligned}
A_1Q =& w_1^{-1}\lambda,\\
QA_2 =& w_1^{-1}(a_3w_1 - \lambda),\\
A_2R =& w_1^{-1}w_2^{-1}(a_3w_1 - \lambda),\\
RA_3 =& w_1^{-1}w_2^{-1}(a_1w_1w_2 - a_3w_1 + \lambda),\\
A_3S =& w_1^{-1}w_2^{-1}w_3^{-1}(a_1w_1w_2 - a_3w_1 + \lambda),\\
SA_1 =& w_1^{-1}w_2^{-1}w_3^{-1}(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda),\\
A_1T =& w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1}(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda),\\
TA_2 =& w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1}(a_3w_1w_2w_3w_4 - a_2w_1w_2w_3 + a_1w_1w_2 - a_3w_1 + \lambda).
\end{aligned}
$$

Then working clockwise from P to U to T,

$$
\begin{aligned}
A_3P =& a_2 - \lambda,\\
UA_3 =& w_6(a_2 - \lambda),\\
A_2U =& a_1 - a_2w_6 + w_6\lambda,\\
TA_2 =& w_5(a_1 - a_2w_6 + w_6\lambda).
\end{aligned}
$$

Equating the two expressions for TA_2 shows that, if the equality is to be independent of λ, the product $w_1w_2w_3w_4w_5w_6$ must equal 1. From this it follows that

$$
w_5 = \frac{a_1w_1w_2 + a_2(1 - w_1w_2w_3) - a_3w_1(1 - w_2w_3w_4)}{a_1w_1w_2w_3w_4}. \tag{5}
$$

Hence w_5 and w_6 may be expressed in terms of the other directions. Given P and the first four directions, points Q, R, S, T are determined, and the five points determine a conic. Independence of λ, used above, ensures that U is also on this conic.

Now the coordinates of the six points may be calculated:

$$
\begin{aligned}
P =& (a_2 - \lambda : 0 : \lambda),\\
Q =& (a_3w_1 - \lambda : \lambda : 0),\\
R =& (0 : a_1w_1w_2 - a_3w_1 + \lambda : a_3w_1 - \lambda),\\
S =& (a_1w_1w_2 - a_3w_1 + \lambda : 0 : a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda),\\
T =& (a_3w_1w_2w_3w_4 - a_2w_1w_2w_3 + a_1w_1w_2 - a_3w_1 + \lambda\\
& : a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda : 0),\\
U =& (0 : a_2 - \lambda : a_3w_1w_2w_3w_4 - a_2w_1w_2w_3 + a_1w_1w_2 - a_3w_1 + \lambda).
\end{aligned}
$$

3.2. These points are on the conic whose equation may be written in the form

$$
\begin{aligned}
&\lambda(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda)(x_1 + x_2 + x_3)^2 \\
&+a_3w_1(a_3w_1w_2w_3w_4 - a_2w_1w_2w_3 + a_1w_1w_2 - a_3w_1 + (1 - w_2w_3w_4)\lambda)x_2^2 \\
&+a_2(a_1w_1w_2 - a_3w_1 + (1 - w_1w_2w_3)\lambda)x_3^2 \\
&-(a_1^2w_1^2w_2^2 + a_3^2w_1^2(1 - w_2w_3w_4) + a_2a_3w_1(1 + w_1w_2w_3) \\
&\qquad - a_3a_1w_1^2w_2(2 - w_2w_3w_4) - a_1a_2w_1^2w_2^2w_3 - (a_2(1 - w_1w_2w_3) \\
&\qquad + a_3w_1(1 - w_2w_3w_4))\lambda)x_2x_3 \\
&-a_2(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - (1 - w_1w_2w_3)\lambda)x_3x_1 \\
&-a_3w_1(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - (1 - w_2w_3w_4)\lambda)x_1x_2 \\
&=0. \qquad (6)
\end{aligned}
$$

The part of (6) not containing the factor $(x_1 + x_2 + x_3)^2$, being linear in λ, represents a pencil of conics. Each of these conics is transformed by a dilatation about its center, induced by the expression containing $(x_1 + x_2 + x_3)^2$. Thus (6) suggests a system of conics analogous to a coaxaloid system of circles. In order to establish the analogy with the Tucker circles, it will be necessary to find a dilatation which transforms every conic by the same ratio of magnification and also transforms (6) into a pencil of conics.

First, if (6) be solved simultaneously with $x_1 + x_2 + x_3 = 0$, it will be found that all terms containing λ vanish. As in the special case, all conics are concurrent at infinity, and it will be shown that all of them are homothetic to each other.

3.3. It is also expected that the centers of the conics will be on one line. When the coordinates $(y_1 : y_2 : y_3)$ of the center are calculated, the results are too long to be displayed here. Suffice it to say that each coordinate is linear in λ, showing that the locus of $(y_1 : y_2 : y_3)$ is a line. If this line is represented by $c_1x_1+c_2x_2+c_3x_3 = 0$, the coefficients, after a large common factor of degree 4 has been removed, may be written as

$$
\begin{aligned}
c_1 &=a_2a_3w_1w_3(w_1 - w_4)(a_2w_2w_3 - a_1w_2 + a_3), \\
c_2 &=a_3w_1(a_3w_3w_4 - a_2w_3 + a_1)(a_1w_1w_2(1 - y) + a_2(1 - x) - a_3w_1(1 - y)), \\
c_3 &=a_2(a_1w_1w_2 - a_3w_1 + a_2)(-a_1(1 - x) + a_2w_3(1 - x) - a_3w_1w_3(1 - y)).
\end{aligned}
$$

For brevity the products $w_1w_2w_3$ and $w_2w_3w_4$ have been represented by the letters x and y respectively.

3.4. As has been seen, addition of any multiple of $(x_1+x_2+x_3)^2$ to the equation of a conic apparently induces a dilatation of the conic about its center. What must now be done, in order to establish an analogy with the system of Tucker circles, is to select a number σ such that the addition of $\sigma(x_1+x_2+x_3)^2$ to (6) dilates every conic by the same ratio ρ and transforms the system of conics into a pencil with two common points besides the two at infinity.

Using a formula for the distance between two points (*e.g.*, [1, p.31]), it may be shown that a dilatation with center $(y_1 : y_2 : y_3)$ sending $(x_1 : x_2 : x_3)$ to $(\overline{x_1} : \overline{x_2} : \overline{x_3})$ with ratio ρ is expressed by $\overline{x_i} \sim y_i + kx_i$, $(i = 1, 2, 3)$, where

$$k = \frac{\pm\rho(y_1 + y_2 + y_3)}{(1 \mp \rho)(x_1 + x_2 + x_3)}$$

or

$$\rho = \frac{\pm(x_1 + x_2 + x_3)}{(y_1 + kx_1) + (y_2 + kx_2) + (y_3 + kx_3)}.$$

In particular, if the conic $\sum a_{ij}x_ix_j = 0$ is dilated about its center $(y_1 : y_2 : y_3)$ with ratio ρ, so that the new equation is

$$\sum a_{ij}x_ix_j + \sigma(x_1 + x_2 + x_3)^2 = 0,$$

then

$$\rho^2 = 1 + \frac{\sigma(y_1 + y_2 + y_3)^2}{\sum a_{ij}y_iy_j}.$$

Here the ambiguous sign is avoided by choosing the a_{ij} so that the denominator of the fraction is positive.

Since it is required that ρ be the same for all conics in (6), it must be free of the parameter λ. For the center $(y_1 : y_2 : y_3)$ of (6), whose coordinates are linear in λ, it may be calculated that $y_1 + y_2 + y_3$ is independent of λ. As for $\sum a_{ij}y_iy_j$, let it first be noted that

$$\begin{aligned} \sum a_{ij}y_jx_i =& (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)x_1 \\ &+ (a_{12}y_1 + a_{22}y_2 + a_{23}y_3)x_2 \\ &+ (a_{13}y_1 + a_{23}y_2 + a_{33}y_3)x_3. \end{aligned}$$

(By convention, $a_{ij} = a_{ji}$). Also, $\sum a_{ij}y_jx_i = 0$ is the equation of the polar line of the center with respect to the conic, but this is the line at infinity $x_1 + x_2 + x_3 = 0$. Therefore the coefficients of x_1, x_2, x_3 in the above equation are all equal, and it follows that

$$\sum a_{ij}y_iy_j = (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)(y_1 + y_2 + y_3),$$

and

$$\rho^2 = 1 + \frac{\sigma(y_1 + y_2 + y_3)}{a_{11}y_1 + a_{12}y_2 + a_{13}y_3}.$$

Since the a_{ij} are quadratic in λ, and the y_i are linear, the denominator of the fraction is at most cubic in λ. Calculation shows that

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 = M(A\lambda^2 + B\lambda + C),$$

in which

$$\begin{aligned}
M =&a_1w_1(a_2w_2w_3 - a_1w_2 + a_3)\cdot\\
&\quad(-a_1^2w_1w_2 + a_2^2w_3 + a_3^2w_1w_3w_4 - a_2a_3w_3(w_1 + w_4)\\
&\qquad + a_3a_1w_1(1 - y) - a_1a_2(1 - x)),\\
A =&a_1w_1w_2 + a_2 + a_3w_1w_2w_3w_4,\\
B =&w_1(a_1^2w_1w_2^2 - a_2^2w_2w_3 - a_3^2w_1w_2w_3w_4 - 2a_2a_3\\
&\qquad - a_3a_1w_1w_2(1 - y) + a_1a_2w_2(1 - x)),\\
C =&a_2a_3w_1^2(a_2w_2w_3 - a_1w_2 + a_3).
\end{aligned}$$

3.5. If the system (6) is to become a pencil of conics, the equation

$$\sum a_{ij}x_ix_j + \sigma(x_1 + x_2 + x_3)^2 = 0$$

must be linear in λ. Since λ^2 appears in (6) as $-\lambda^2(x_1+x_x+x_3)^2$, this will vanish only if the coefficient of λ^2 in σ is 1. Therefore, to eliminate λ from the fractional part of ρ^2, it follows that

$$\sigma = \lambda^2 + \frac{B}{A}\lambda + \frac{C}{A}.$$

With this value of σ, if $\sigma(x_1 + x_2 + x_3)^2$ be added to (6), the equation becomes

$$\begin{aligned}
&\lambda(-a_2a_3w_1(1 - xy)(x_1 + x_2 + x_3)^2 + (a_1w_1w_2 + a_2 + a_3w_1y)\cdot\\
&\quad(a_3w_1(1 - y)x_2^2 + a_2(1 - x)x_3^2 + (a_2(1 - x) + a_3w_1(1 - y))x_2x_3\\
&\quad + a_2(1 - x)x_3x_1 + a_3w_1(1 - y)x_1x_2))\\
&+a_2a_3w_1^2(a_2w_2w_3 - a_1w_2 + a_3)(x_1 + x_2 + x_3)^2\\
&+(a_1w_1w_2 + a_2 + a_3w_1y)(a_3w_1(a_1w_1w_2 - a_2x - a_3w_1(1 - y))x_2^2\\
&+a_2w_1(a_1w_2 - a_3)x_3^2\\
&-(a_1^2w_1^2w_2^2 + a_3^2w_1^2(1 - y) + a_2a_3w_1(1 + x)\\
&\qquad - a_3a_1w_1^2w_2(2 - y) - a_1a_2w_1w_2x)x_2x_3\\
&+a_2(a_1w_1w_2 - a_2x - a_3w_1)x_3x_1\\
&+a_3w_1(a_1w_1w_2 - a_2x - a_3w_1)x_1x_2)\\
&=0. \qquad (7)
\end{aligned}$$

Since (7) is linear in λ, it represents a pencil of conics. These conics should have four points in common, of which two are known to be at infinity. In order to facilitate finding the other two points, it is noted that a pencil contains three degenerate conics, each one consisting of a line through two of the common points, and the line of the other two points. In this pencil the line at infinity and the line of the other two common points comprise one such degenerate conic. Its equation may be given by setting equal to zero the product of $x_1 + x_2 + x_3$ and a second linear factor. Since it is known that the coefficient of λ vanishes at infinity, the conic represented by $\lambda = \infty$ in (7) must be the required one. The coefficient of λ does indeed factor as follows:

$$\begin{aligned}&(x_1 + x_2 + x_3)(-a_2a_3w_1(1 - xy)x_1\\&\quad + (a_3w_1(1 - y)(a_1w_1w_2 + a_2 + a_3w_1y) - a_2a_3w_1(1 - xy))x_2\\&\quad + (a_2(1 - x)(a_1w_1w_2 + a_2 + a_3w_1y) - a_2a_3w_1(1 - xy))x_3).\end{aligned}$$

Therefore the second linear factor equated to zero must represent the line through the other two fixed points of (7).

3.6. These points may be found as the intersection of this line and any other conic in the system, for example, the conic given by $\lambda = 0$. To solve simultaneously the equations of the line and the conic, x_1 is eliminated, reducing the calculation to

$$a_3^2w_1w_2w_3w_4x_2^2 - a_2a_3(1 + w_1w_2^2w_3^2w_4)x_2x_3 + a_2^2w_2w_3x_3^2 = 0$$

or

$$(a_3x_2 - a_2w_2w_3x_3)(a_3w_1w_2w_3w_4x_2 - a_2x_3) = 0.$$

Therefore,

$$\frac{x_2}{x_3} = \frac{a_2w_2w_3}{a_3} \quad \text{or} \quad \frac{a_2}{a_3w_1w_2w_3w_4}.$$

The first solution gives the point

$$\Lambda = (a_1w_2w_3w_4w_5 : a_2w_2w_3 : a_3)$$

and the second solution gives

$$\Lambda' = (a_1w_1w_2 : a_2 : a_3w_1w_2w_3w_4).$$

Thus the dilatation of every conic of (6) about its center with ratio ρ transforms (6) into pencil (7) with common points Λ and Λ'.

3.7. Returning to the question of whether the conics of (6) are all homothetic to each other, this was settled in the case of parabolas. As for hyperbolas, it was found that they all have respectively parallel asymptotes, but a hyperbola could be enclosed in the acute sectors formed by the asymptotes, or in the obtuse sectors. However, when (6) is transformed to (7), there are at least two hyperbolas in the pencil that are homothetic. Since the equation of any hyperbola in the pencil may be expressed as a linear combination of the equations of these two homothetic ones, it follows that all hyperbolas in the pencil, and therefore in system (6), are homothetic to each other. A similar argument shows that, if (6) consists of ellipses, they must all be homothetic. Figure 5 shows a system (6) of ellipses, with one hexagon left in place. In Figure 6 the same system has been transformed into a pencil with two common points. Figure 7 shows two hyperbolas of a system (6), together with their hexagons. The related pencil is not shown.

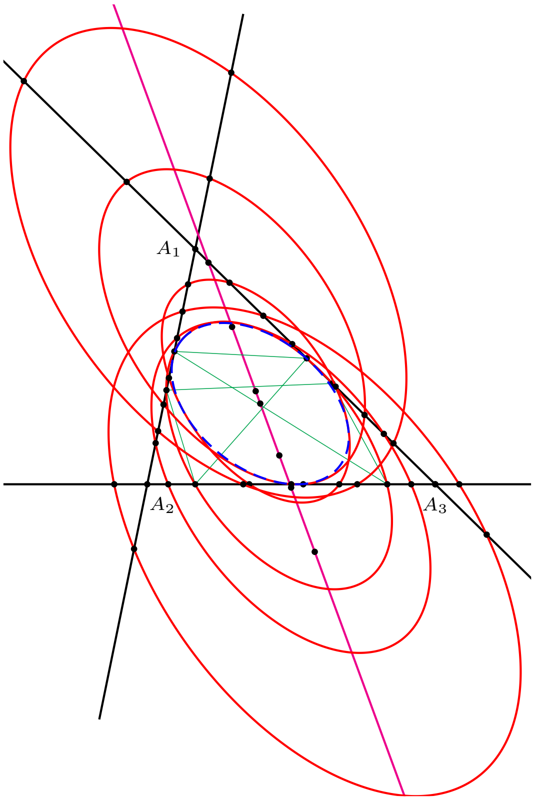

Figure 5

3.8. In the barycentric coordinate system, the midpoint $(v_1 : v_2 : v_3)$ of $(x_1 : x_2 : x_3)$ and $(y_1 : y_2 : y_3)$ is given by

$$v_i \sim \frac{x_i}{x_1 + x_2 + x_3} + \frac{y_i}{y_1 + y_2 + y_3}, \quad i = 1, 2, 3.$$

Thus it may be shown that the coordinates of the midpoint of $\Lambda\Lambda'$ are

$$(2a_1 w_1 w_2 + a_2(1 - x) - a_3 w_1(1 - y) : a_2(1 + x) : a_3 w_1(1 + y)).$$

This point is on the line of centers of (6), expressed earlier as

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0,$$

so the segment $\Lambda\Lambda'$ is bisected by the line of centers. However, it is not the perpendicular bisector unless (6) consists of circles. This case has already been disposed of, because if a circle cuts the sidelines of $A_1A_2A_3$, PQ and ST must be antiparallel to each other, as must QR and TU, and RS and UP. This would mean that (6) is a Third system.

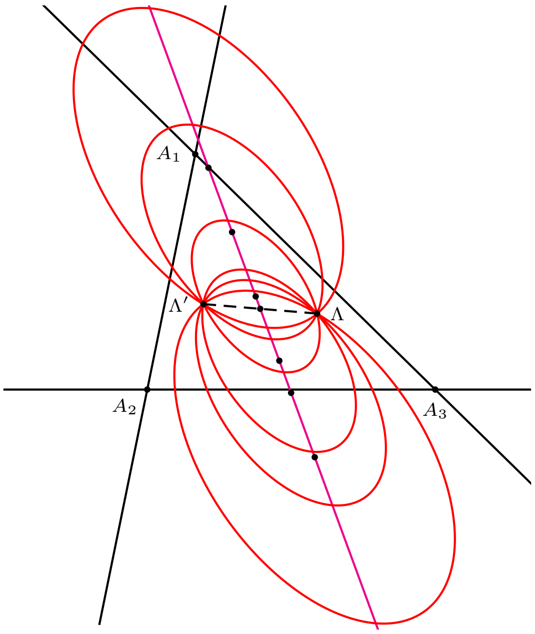

Figure 6

In system (6) the lines PQ, RS, TU are concurrent for a unique value of λ, which has been calculated but will not be written here. The point of concurrence is

$$\left(\frac{1}{-a_1w_1w_2+a_2x+a_3w_1}:\frac{1}{a_1w_1w_2-a_2x+a_3w_1}:\frac{1}{a_1w_1w_2+a_2x-a_3w_1}\right),$$

which is a generalization of (2). The same point is obtained when QR, ST. UP are concurrent. It will also be written as $\left(\frac{1}{F_1}:\frac{1}{F_2}:\frac{1}{F_3}\right)$.

3.9. System (6) has an envelope which may be found by writing (6) as a quadratic equation in λ. Setting its discriminant equal to zero gives an equation of the envelope. The discriminant contains the factor $(x_1+x_2+x_3)^2$, which may be deleted, leaving

$$\sum F_i^2x_i^2-2F_jF_kx_jx_k=0.$$

This is an equation of the conic which touches the sidelines of $A_1A_2A_3$ at $L_1=(0:F_3:F_2)$, $L_2=(F_3:0:F_1)$, and $L_3=(F_2:F_1:0)$. The cevians A_iL_i are concurrent at $\left(\frac{1}{F_1}:\frac{1}{F_2}:\frac{1}{F_3}\right)$.

The center of the envelope is the midpoint of $\Lambda\Lambda'$, but Λ and Λ' are not foci unless they are isogonal conjugates. This happens when $(w_1w_2)(w_2w_3w_4w_5)=(1)(w_2w_3)=(w_1w_2w_3w_4)(1)$, for which the solution is

$$w_1w_4=w_2w_5=w_3w_6=1.$$

Since this defines a Third system, it follows that Λ and Λ' are isogonal conjugates (and foci of the envelope of (6)) if and only if the conics are circles.

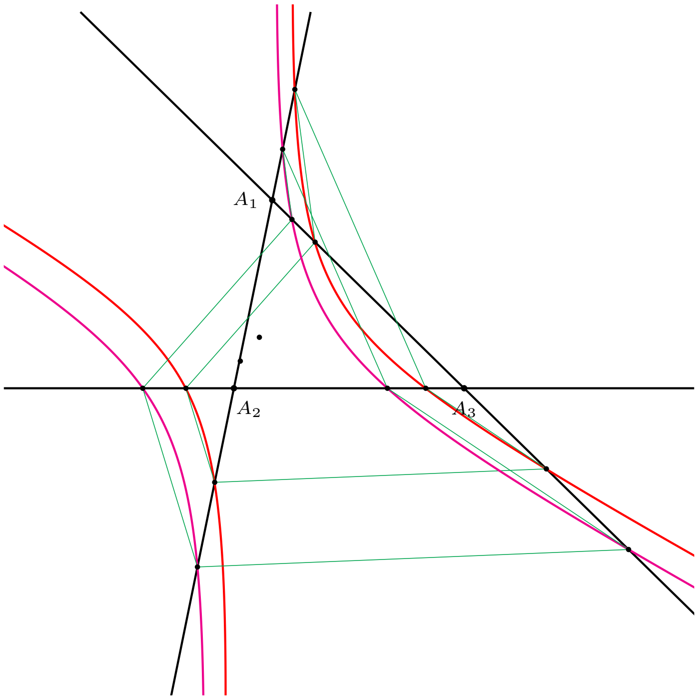

Figure 7

4. The parabolic case

There remains the question of whether (6) can be a system of parabolas. This is because the dilatations used above were made from the centers of the conics, whereas the centers of parabolas may be regarded as being at infinity. If the theory still holds true, the dilatations would have to be translations. That such cases actually exist may be demonstrated by the following example.

Let the triangle have sides $a_1 = 4$, $a_2 = 2$, $a_3 = 3$, and let

$$w_1 = \frac{2}{3}, \quad w_2 = \frac{3}{4}, \quad w_3 = \frac{1}{2}, \quad w_4 = \frac{2}{3}, \quad w_5 = 3, \quad w_6 = 2.$$

Substitution of these values in (6) gives the equation (after multiplication by 2)

$$\begin{aligned} &\lambda(1-2\lambda)(x_1+x_2+x_3)^2 + 3\lambda x_2^2 + 3\lambda x_3^2 \\ &+2(3\lambda-4)x_2x_3 + (3\lambda-2)x_3x_1 + (3\lambda-2)x_1x_2 = 0. \end{aligned} \tag{8}$$

To verify that this is a system of parabolas, solve (8) simultaneously with $x_1 + x_2 + x_3 = 0$, and elimination of x_1 gives the double solution $x_2 = x_3$. This shows that for every λ the conic is tangent to the line at infinity at the point $(-2 : 1 : 1)$. Hence, every nondegenerate conic in the system is a parabola, and all are homothetic to each other. (See Figure 8).

The formulae for σ gives the value $\left(\lambda - \frac{2}{3}\right)^2$, but (8) was obtained after multiplication by 2. Therefore $2\left(\lambda - \frac{2}{3}\right)^2 (x_1 + x_2 + x_3)^2$ is added to (8), yielding the

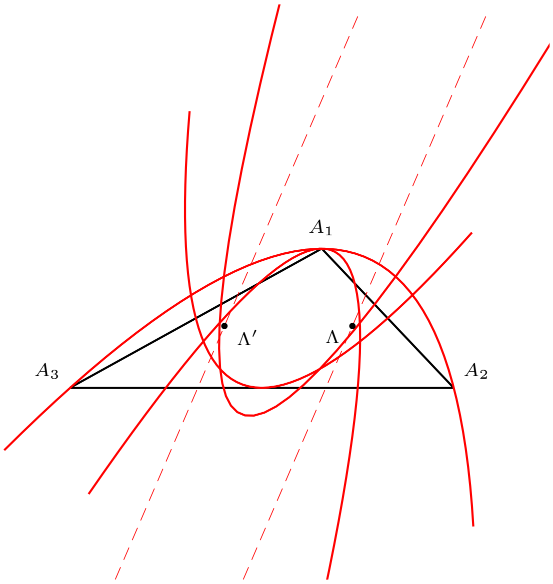

Figure 8. A system of parabolas

equation

$$(8-15\lambda)x_1^2+4(2+3\lambda)(x_2^2+x_3^2)-8(7-3\lambda)x_2x_3-(2+3\lambda)(x_3x_1+x_1x_2)=0,$$

which is linear in λ and represents a pencil of parabolas. The parabola $\lambda=\infty$ is found by using only terms containing λ, which gives the equation

$$-15x_1^2+12x_2^2+12x_3^2+24x_2x_3-3x_3x_1-3x_1x_2=0$$

or

$$3(x_1+x_2+x_3)(-5x_1+4x_2+4x_3)=0.$$

Thus it is the degenerate conic consisting of the line at infinity and the line $-5x_1+4x_2+4x_3=0$. Calculation shows that this line intersects every parabola of the pencil at $\Lambda(4:1:4)$ and $\Lambda'(4:4:1)$. (See Figure 9). The parallel dashed lines in both figures form the degenerate parabola $\lambda=\frac{2}{3}$, which is invariant under the translation which transformed the system into a pencil.

Finally, since all of the "centers" coincide, this is another exception to the rule that the line of centers of (6) bisects the segment $\Lambda\Lambda'$.

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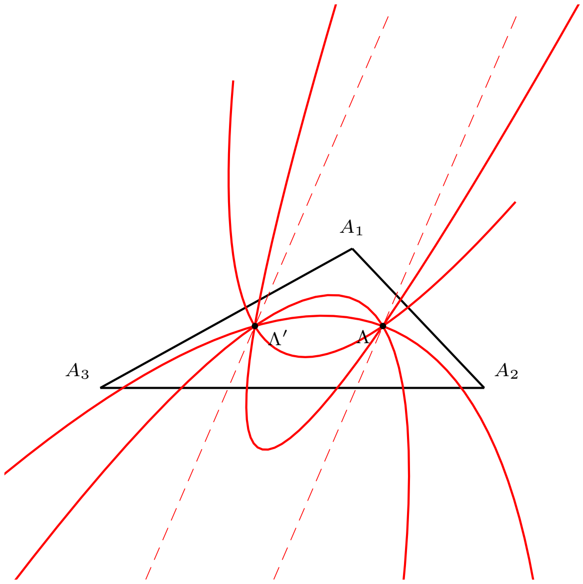

Figure 9. A pencil of parabolas

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