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Geometry and Group Structures of Some Cubics

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Abstract. We review the group structure of a cubic in the projective complex plane and give group theoretic formulations of some geometric properties of a cubic. Then, we apply them to pivotal isocubics, in particular to the cubics of Thomson, Darboux and Lucas. We use the group structure to identify different transformations of cubics. We also characterize equivalence of cubics in terms of the Salmon cross ratio.

1. The group structure of a cubic

Let Γ be a nonsingular cubic curve in the complex projective plane, *i.e.*, Γ has no cusp and no node. It is well known that Γ has a group structure, which does not depend on the choice of a neutral element O on the cubic. In other words, the group structures on Γ for various choices of the neutral elements are isomorphic.

If P and Q are points of a cubic Γ, we denote by $P \cdot Q$ the third intersection of the line PQ with Γ. In particular, $P \cdot P := P_t$ is the *tangential* of P, the second intersection of Γ with the tangent at P.

Proposition 1. *The operation $\cdot$ is commutative but not associative. For P, Q, R on Γ,*

(1) $(P \cdot Q) \cdot P = Q$,
(2) $P \cdot Q = R \cdot Q \Longleftrightarrow P = R$,
(3) $P \cdot Q = R \Longleftrightarrow P = R \cdot Q$.

Convention: When we write $P \cdot Q \cdot R$, we mean $(P \cdot Q) \cdot R$.

We choose a point O on Γ as the neutral point, [1] and define a group structure $+$ on Γ by

$$P + Q = (P \cdot Q) \cdot O.$$

We call the tangential of O, the point $N = O_t = O \cdot O$, the *constant point* of Γ. Note that $-N = N_t$, since $N + N_t = N \cdot N_t \cdot O = N \cdot O = O$.

We begin with a fundamental result whose proof can be found in [4, p.392].

Theorem 2. *$3k$ points P_i, $1 \leq i \leq 3k$, of a cubic Γ are on a curve of order k if and only if $\sum P_i = kN$.*

For $k = 1, 2, 3$, we have the following corollary.

Corollary 3. *Let P, Q, R, S, T, U, V, W, X be points of Γ.*

(1) *P, Q, R are collinear if and only if $P + Q + R = N$.*
(2) *P, Q, R, S, T, U are on a conic if and only if $P+Q+R+S+T+U = 2N$.*

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[1] O is not necessarily an inflexion point (a flex).

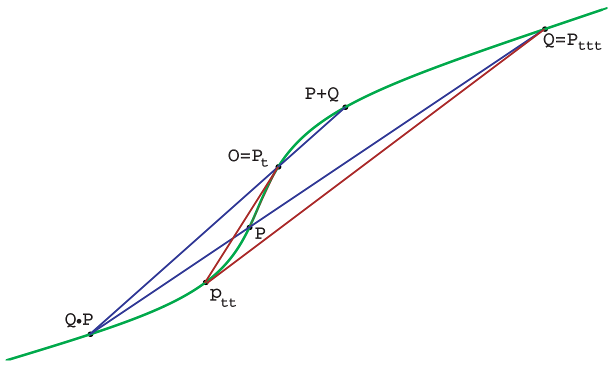

Figure 1. The first three tangentials of P and $P+Q$

(3) *P, Q, R, S, T, U, V, W, X are on a cubic if and only if* $P+Q+R+S+T+U+V+W+X=3N$.

Remark. The case $k=2$ is equivalent to the following property.

Geometric formulation	Group theoretic formulation
Let P, Q, R, S, T, U be six points of a cubic Γ, and let $X=P\cdot Q$, $Y=R\cdot S$, $Z=T\cdot U$, then P, Q, R, S, T, U are on a conic if and only if X, Y, Z are collinear.	Let P, Q, R, S, T, U be six points of a cubic Γ, and let $P+Q+X=N$, $R+S+Y=N$, $T+U+Z=N$, then $P+Q+R+S+T+U=2N$ if and only if $X+Y+Z=N$.

A geometrical proof is given [8, p.135]; an algebraic proof is a straightforward calculation.

We can do normal algebraic calculations in the group, but have to be careful to torsion points: for example $2P=O$ does not imply $P=O$. The group of Γ has non zero torsion points, i.e, points with the property $kP=O$, for $P\neq O$. Indeed the equation $kX=Q$ has k^2 (complex) solutions for the point X. See [10, 17].

The tangential P_t of P is $N-2P$, since P, P, and P_t are collinear. The *second tangential* P_{tt} of P is $N-2(N-2P)=-N+4P$. The *third tangential* is $N-2(-N+4P)=3N-8P$.

2. A sample of theorems on cubics

We give a sample of theorems on cubics, in both geometric and group-theoretic formulations. Most of the theorems can be found in [8, p.135]. In the following table, all points are on a cubic Γ. A point $P\in\Gamma$ is a *sextatic* point if there is a conic through P with contact of order 6 with Γ at P.

	Geometric formulation	Group theoretic formulation
1	P and Q have the same tangential.	$2P = 2Q$ or $2(P - Q) = O$
2	There are four tangents from P.	$2X + P = N$ has four solutions
3	P is a flex	$3P = N$
4	Γ has nine flexes	$3P = N$ has nine solutions
5	If P and Q are flexes, then $R = P \cdot Q$ is another flex. If $P \neq Q$, then $R \neq P,\ Q$.	$3P = N, 3Q = N$, and $P + Q + R = N$ $\Longrightarrow 3R = N$.
6	Let P_1, P_2, P_3 and P_4 be fixed on Γ. If a variable conic intersects Γ at $P_1 \ldots, P_6$, then the line P_5P_6 passes through a fixed point Q on Γ, which we call the *coresidual* of P_1, P_2, P_3, P_4.	$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 2N$ and $P_5 + P_6 + Q = N$ $\Longrightarrow Q = -N + P_1 + P_2 + P_3 + P_4$, which is fixed.
7	If a conic intersects Γ at $P_1, \ldots P_6$, then the tangentials $Q_1, \ldots, Q_6$ are on another conic	$\sum P_i = 2N, 2P_i + Q_i = N$ for $i = 1, \ldots, 6$ $\Longrightarrow \sum Q_i = 2N$.
8	Let Ω be a conic *tritangent* to Γ at P, Q, R, and let Ψ be another conic which intersects Γ at P, Q, R, P', Q', R', then there exists a conic Λ tangent to Γ at P', Q', R'.	$2P + 2Q + 2R = 2N$ and $P + Q + R + P' + Q' + R' = 2N$ $\Longrightarrow 2P' + 2Q' + 2R' = 2N$.
9	A conic Ω is tritangent to Γ at P, Q, R if and only if the tangentials P', Q', R' of P, Q, R are collinear.	For $2P + P' = N$, $2Q + Q' = N$, and $2R + R' = N$, $2P + 2Q + 2R = 2N$ $\Longleftrightarrow P' + Q' + R' = N$.
10	If Q, R, S are given points, there exist 9 points X such that a conic *osculates* at X and passes through Q, R, S	The equation $3X + Q + R + S = 2N$ has nine solutions.
11	P is sextatic if and only if the tangent at P contains a flex Q different from P.	For $2P + Q = N$, $6P = 2N \Longleftrightarrow 3Q = N$.
12	P is sextatic if and only if P is the tangential of a flex Q.	$6P = 2N \Longleftrightarrow$ $2Q + P = N$ and $3Q = N$.
13	There are 27 sextatic points on a cubic.	$6P = 2N$ has 36 solutions, nine are the flexes, the others 27 are the sextatic points.
14	If P and Q are sextatic, then $R = P \cdot Q$ is sextatic.	$6P = 2N$, $6Q = 2N$ and $P + Q + R = N$ $\Longrightarrow 6R = 2N$.

Remarks. The coresidual in (6) is called the *gegenüberliegende Punkt* in [8, p.140].

3. The group structure of a pivotal isocubic

Let $P \mapsto P^*$ be a given isoconjugation in the plane of the triangle ABC (with trilinear coordinates). See, for example, [5]. For example, $P(x : y : z) \mapsto P^*(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ is the isogonal transformation and $P(x : y : z) \mapsto (\frac{1}{a^2x} : \frac{1}{b^2y} : \frac{1}{c^2z})$ is the isotomic transformation. We shall also consider the notion of cevian quotient. For any two points P and Q, the cevian triangle of P and the precevian triangle of

Q are always perspective. We call their perspector the *cevian quotient* P/Q. See [11].

Let Γ be a pivotal isocubic with pivot F. See, for example, [6, 7, 14]. Take the pivot F for the neutral element O of the group. The constant point is $N = F_t$.

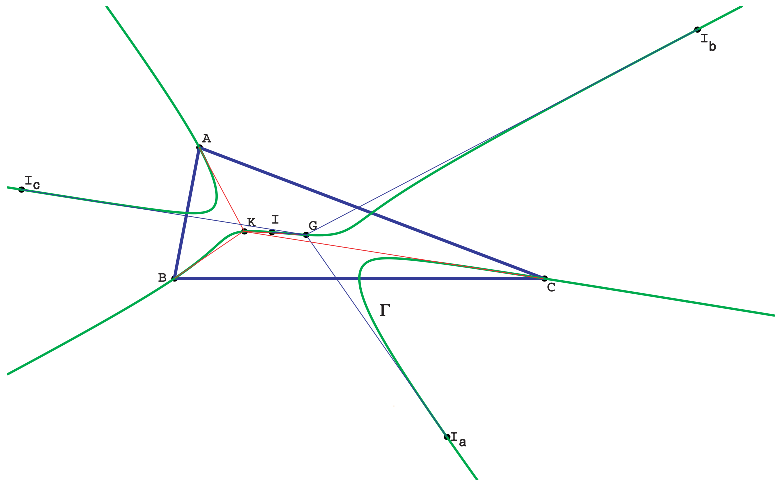

Figure 2. Two tangential quadruples on the Thomson cubic

Definition. Four points of Γ form a *tangential quadruple* if they have the same tangential point.

Theorem 4. *Consider the group structure on a pivotal isocubic with the pivot F as neutral element. The constant point is $N = F_t$.*

(1) $P \cdot P^* = F$, $P \cdot F = P^*$, $P^* \cdot F = P$.
(2) $F_t = F^*$.
(3) $P + P^* = F_t$.
(4) $P + Q = (P \cdot Q)^*$ *or* $P \cdot Q = (P + Q)^*$.
(5) P, Q, R *are collinear if and only if* $P + Q + R = F_t$.
(6) $-P = P \cdot F_t$.
(7) $-P = F/P$.
(8) *If* (P, Q, R, S) *is a tangential quadruple then* (P^*, Q^*, R^*, S^*) *is also a tangential quadruple.*
(9) *Every tangential quadruple is of the form* $(P, P + A, P + B, P + C)$.
(10) A, B, C *are points of order 2,* i.e., $2A = 2B = 2C = F$.

Proof. (1) F is the pivot, so P, P^* and F are collinear.

(2) Put $P = F$ in (1).

(3) $P + P^* = (P \cdot P^*) \cdot F = F \cdot F = F_t$.

(4) $P + Q = (P \cdot Q) \cdot F = (P \cdot Q)^*$. (use (1))

(5) This is Corollary 3.

(6) $P + (P \cdot F_t) = (P \cdot (P \cdot F_t)) \cdot F = F_t \cdot F = F$.

(7) If the pivot F has trilinear coordinates $(u:v:w)$ and $P(x:y:z)$, then the Cevian quotient F/P is the point

$$(x(-vwx+uwy+uvz):\ y(vwx-uwy+uvz):\ z(vwx+uwy-uvz)).$$

We can verify that it is on Γ and is collinear with P and F_t.

(8) We have to prove that, if P and Q have a common tangential T, then P^* and Q^* have a common tangential U. Let U be the tangential of P^*, then (5) and (2) give

$$U+2P^*=F_t=F^*.$$

Since F, P, P^* are collinear, and so are F, Q, Q^*, we have

$$P+P^*=F^* \quad \text{and} \quad Q+Q^*=F^*.$$

Since T is the common tangential of P and Q,

$$2P+T=F^* \quad \text{and} \quad 2Q+T=F^*.$$

From these,

$$\begin{aligned} U+2Q^* =&(F^*-2P^*)+2Q^* \\ =&F^*-2(F^*-P)+2(F^*-Q) \\ =&F^*+2P-2Q \\ =&F^*+F^*-T-F^*+T \\ =&F^*, \end{aligned}$$

and U is the tangential of Q^* too.

(9) We have to prove that, if P is on the cubic, P and $P+A$ have the same tangential. Let Q and Q_a be the tangential of P and $P+A$ respectively. By property (3), $P+P+Q=F^*$ and $(P+A)+(P+A)+Q_a=F^*$. Hence $Q=Q_a \iff 2A=0 \iff A=-A$. By properties (6) and (2), $-A=A\cdot F^*$, hence we have to prove that $A=A\cdot F^*$, i.e. the tangential of A is F^*. The equation of the tangent to the cubic at A is $r^2vy=q^2wz$, and $F^*(p^2vw:q^2uw:r^2uv)$ is clearly on this line. But F^* is on Γ. Hence it is the tangential point of A.

(10) $2A=A+A=A\cdot A=A_t^*=F^{**}=F$. $\square$

A consequence of (10) is that the cubic is not connected. See, for example, [10, p.20].

4. The Thomson, Darboux and Lucas cubics

These well-known pivotal cubics have for pivots G (centroid), L (de Longchamps point) and K_+ (isotomic of the orthocenter H). Thomson and Darboux are isogonal cubics and Lucas is an isotomic one. We study the subgroups generated by the points G, I, A, B, C for Thomson, L, I', A, B, C for Darboux and K_+, N_o, A, B, C for Lucas. For a generic triangle, [2] these groups are isomorphic to $\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2$.

[2]This may be false for some particular triangles. For example, if ABC has a right angle at A, then $H=A$ and for Thomson, $H=4I$.

Notation. For each point P, we denote by P_a, P_b, P_c the points $P+A$, $P+B$, $P+C$ respectively. We use the notations of [12] for triangle centers, but adopt the following for the common ones.

G	centroid
K	symmedian (Lemoine) point
H	orthocenter
O	circumcenter
I	incenter; I_a, I_b, I_c are the excenters
L	de Longchamps point
M	Mittenpunkt
G_o	Gergonne point
N_o	Nagel point

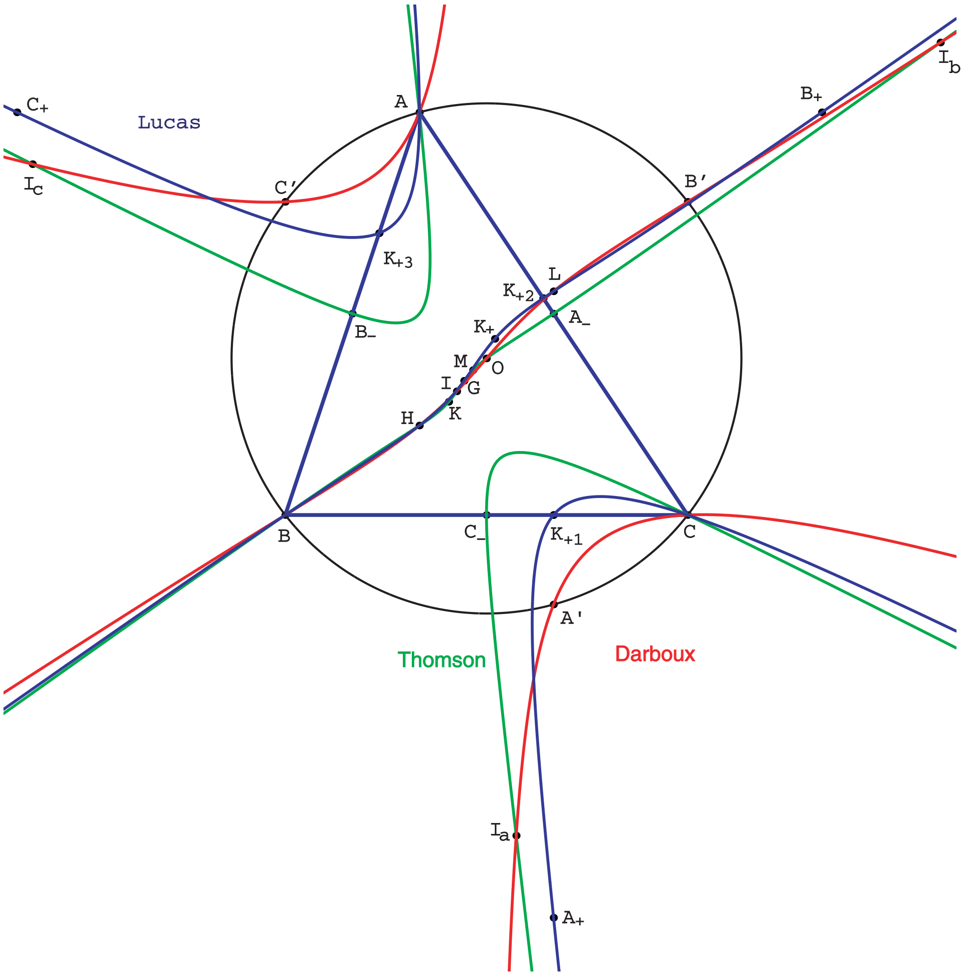

Figure 3. The Thomson, Darboux and Lucas cubics

In the following table, the lines represent the $\mathbb{Z}$-part, and the columns the $\mathbb{Z}_2 \times \mathbb{Z}_2$-part. The last column give the tangential point of the tangential quadruple of the corresponding line. The line number 0 is the subgroup generated by the pivot and A, B, C. It is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4.1. *The Thomson cubic.*

	P	P_a	P_b	P_c	tangential
-6	H_t	H_{ta}	H_{tb}	H_{tc}	
-5	$-X_{282}$	$-X_{282a}$	$-X_{282b}$	$-X_{282c}$	
-4	O_t^*	O_{ta}^*	O_{tb}^*	O_{tc}^*	
-3	X_{223}	X_{223a}	X_{223b}	X_{223c}	
-2	O	A_-	B_-	C_-	O_t
-1	M	M_a	M_b	M_c	H
0	G	A	B	C	K
1	I	I_a	I_b	I_c	G
2	K	A_-	B_-	C_-	O
3	M^*	M_a^*	M_b^*	M_c^*	O_t^*
4	H	H_a	H_b	H_c	H_t
5	X_{282}	X_{282a}	X_{282b}	X_{282c}	
6	O_t	O_{ta}	O_{tb}	O_{tc}	O_{tt}

(1) Neutral = pivot = G = centroid.
(2) Constant $= G_t = G^* = K$.
(3) Three points are collinear if and only if their sum is 2.
(4) Examples of calculation:
   (a) $I + K = (I \cdot K) \cdot G = M \cdot G = M^*$.
   (b) $A + A = (A \cdot A) \cdot G = K \cdot G = G$.
   (c) To find the intersection X of the line OM with the cubic, we have to solve the equation $x + (-2) + (-1) = 2$. Hence, $x = 5$ and $X = X_{282}$.
(5) A_-, B_-, C_- are the midpoints of the sides of ABC, diagonal triangle of $GABC$.
(6) A^-, B^-, C^- are the midpoints of the altitudes of ABC, diagonal triangle of $KA_-B_-C_-$.
(7) O_t is the isoconjugate of the circumcenter O relative to the pencil of conics through the points K, A_-, B_-, C_-.
(8) O_{tt} is the isoconjugate of O_t relative to the pencil of conics through the points O, A^-, B^-, C^-.
(9) $O_{ta}O_{tb}O_{tc}$ is the diagonal triangle of $OA^-B^-C^-$.
(10) $H_a = A^{-*} = OA \cap A^-G = B^-C \cap C^-B$.
(11) $X_{223} = -(M^*)$ is the third intersection of the line IH and Γ. (Proof: $I + H + X_{223} = 1 + (-3) + 4 = 2 =$ constant).
(12) If a point X has the line number x, then the points X^*, X_t and G/X have line numbers $2 - x$, $2 - 2x$ and $-x$.

4.2. *The Darboux cubic.*

	P	P_a	P_b	P_c	tangential
-6	L_t^*				
-5	$-I^{'*}$				
-4	$-H$				
-3	$I^{'*'*'}$	$I_a^{'*'*'}$	$I_b^{'*'*'}$	$I_c^{'*'*'}$	
-2	$L^{*'}$	L'_a	L'_b	L'_c	
-1	$I^{'*'}$	$I_a^{'*'}$	$I_b^{'*'}$	$I_c^{'*'}$	
0	L	A	B	C	L^*
1	I'	I'_a	I'_b	I'_c	H
2	O	$H_{1\infty}$	$H_{2\infty}$	$H_{3\infty}$	O
3	I	I_a	I_b	I_c	L
4	H	A'	B'	C'	$L_{*'}$
5	$I^{'*}$	$I_a^{'*}$	$I_b^{'*}$	$I_c^{'*}$	
6	L^*	L_1	L_2	L_3	L_t^*
7	$I^{'*'*}$	$I_a^{'*'*}$	$I_b^{'*'*}$	$I_c^{'*'*}$	
8	$L^{*'*}$				

(1) Neutral = pivot = L = de Longchamps point = symmetric of H relative to O; constant point = L^*.
(2) Three points are collinear if and only if their sum is 6.
(3) L_1, L_2, L_3 = Cevian points of L.
(4) $H_{1\infty}$ = infinite point in the direction of the altitude AH.
(5) P' is the symmetric of P relative to O. (Symmetry relative to line 2)
(6) $P^{'*}$ gives the translation of $+2$ and $P^{*'}$ of -2. Three points are collinear if and only if their sum is 6.
(7) If a point X has the line number x, then the points X^*, X_t, X' and G/X have line numbers $6-x$, $6-2x$, $4-x$ and $-x$.

4.3. *The Lucas cubic.*

	P	P_a	P_b	P_c	tangential
-4	K_{+t}				
-3	$-G_o$	$-G_a$	$-G_b$	$-G_c$	
-2	L	L_a	L_b	L_c	X_{1032}
-1	X_{329}	X_{329a}	X_{329b}	X_{329c}	L^r
0	K_+	A	B	C	H
1	N_o	N_a	N_b	N_c	G
2	G	A_+	B_+	C_+	K_+
3	G_o	G_a	G_b	G_c	L
4	H	K_{+1}	K_{+2}	K_{+3}	K_{+t}
5	X_{189}	X_{189a}	X_{189b}	X_{189c}	
6	L^r	L_a^r	L_b^r	L_c^r	
7	X_{1034}	X_{1034a}	X_{1034b}	X_{1034c}	
8	X_{1032}				

(1) Neutral = K_+ = Lemoine point of the precevian triangle $A_+B_+C_+$ of ABC = isotomic of H; Constant point = H. Three points are collinear if and only if their sum is 4.
(2) P^r = isotomic of P (symmetry relative to line 2).
(3) K_{+1}, K_{+2}, K_{+3} = cevian points of K_+ = intersections of Lucas cubic with the sides of ABC.
(4) X_{329} = intersection of the lines N_oH and G_oG with the cubic.
(5) If a point X has line number x, then the points X^*, X_t and G/X have line numbers $4-x$, $4-2x$ and $-x$.
(6) $X_{329}^r = X_{189}$.

5. Transformations of pivotal isocubics

We present here some general results without proofs. See [16, 15, 9, 4].

5.1. *Salmon cross ratio.* The Salmon cross ratio of a cubic is the cross ratio of the four tangents issued from a point P of Γ. It is defined up to permutations of the tangents. We shall therefore take it to be a set of the form

$$\left\{\lambda,\ \lambda-1,\ \frac{1}{\lambda},\ \frac{1}{\lambda-1},\ \frac{\lambda}{\lambda-1},\ \frac{\lambda-1}{\lambda}\right\},$$

since if λ is a Salmon cross ratio, then we obtain the remaining five values of permutation of the tangents.

A cubic Γ is *harmonic* if $\lambda = -1$; it is *equiharmonic* if λ satisfies $\lambda^2-\lambda+1 = 0$.

The Salmon cross ratio is independent of the choice of P.

5.2. *Birational equivalence.* A transformation [9] of Γ is *birational* if the transformation and its inverse are given by rational functions of the coordinates. [3] Two cubics Γ_1 and Γ_2 are equivalent if there is a birational transformation $\Gamma_1 \to \Gamma_2$.

Theorem 5. *A birational transformation of a cubic Γ onto itself induces a transformation of its group of the form $x \mapsto ux + k$, where*
(1) $u^2 = 1$ *for a general cubic,*
(2) $u^4 = 1$ *for a harmonic cubic, and*
(3) $u^6 = 1$ *for an equiharmonic cubic.*

Theorem 6. *Two equivalent cubics have isomorphic groups.*

Examples:
1) The groups of the cubics of Darboux, Thomson and Lucas are isomorphic.
2) The transformation that associate to a point its tangential is given by $X \mapsto N - 2X$ and is not birational.

Theorem 7. *Two cubics Γ_1 and Γ_2 are equivalent if and only if their Salmon cross ratios are equal.*

[3]Cautions: Two different transformations of the projective plane may induce the same transformation on curves. see [15].

If the isoconjugation has fixed point $(p : q : r)$, it is easy to prove the following result:

Theorem 8. *A pivotal isocubic of pivot* $(u : v : w)$ *has Salmon cross ratio*

$$\frac{q^2(r^2u^2 - p^2w^2)}{r^2(q^2u^2 - p^2v^2)}.$$

For example, the cubics of Darboux, Thomson, Lucas all have Salmon cross ratio

$$\frac{b^2(a^2 - c^2)}{c^2(a^2 - b^2)}.$$

Are Thomson, Darboux and Lucas the only equivalent pivotal cubics? No! Here is a counter-example. Take the isoconjugation with fixed point X_{63}. The pivotal isocubic of pivot X_{69} (the same as Lucas) is equivalent to Thomson.

6. Examples of birational transformations of cubics

We give now a list of birational transformations, with the corresponding effects on the lines of the group table. Recall that N is the tangential of the pivot, *i.e.*, the constant point.

6.1. *Projection:* $\Gamma \to \Gamma$. Let $P \in \Gamma$. A projection of Γ on itself from P gives a transformation $X \mapsto X'$ so that P, X, X' are collinear:

$$x \mapsto n - p - x.$$

6.2. *Cevian quotient:* $\Gamma \to \Gamma$. Let F be the pivot of Γ, then the involution $X \mapsto F/X$ gives the transformation: $x \mapsto -x$.

6.3. *Isoconjugation:* $\Gamma \to \Gamma$. Since F, X, X^* are collinear, the isoconjugation is a projection from the pivot $F : x \mapsto n - x$.

6.4. *Pinkernell's quadratic transformations.* We recall the definition of the $d-$pedal cubics Γ_d and of the $d-$cevian cubics Δ_d. If P has *absolute* trilinear coordinates (x, y, z), then define P_A, P_B, P_C on the perpendiculars from P to the sides such that $PP_A = dx$, etc. The locus of P for which $P_AP_BP_C$ is perspective to ABC is a cubic Γ_d, and the locus of the perspector is another cubic Δ_d. Hence we have a birational transformation $f_d : \Gamma_d \to \Delta_d$.

The $d-$pedal is different from the $(-d)$-pedal, but the d-cevian is the same as the $(-d)$-cevian.

For example: $\Gamma_1 =$ Darboux , $\Gamma_{-1} =$ Thomson, and $\Delta_1 =$ Lucas.

Let L_d be the pivot of Γ_d and X on Γ_d. Since L_d, X and $f_d(X)$ are collinear we can identify f_d as a projection of Γ_d to Δ_d from the pivot L_d.

These transformations are birational. Hence the groups of the cubics Γ_d, Γ_{-d} and Δ_d are isomorphic.

For $d = 1$, X and $f_d(X)$ are on the same line in the group table: $x \mapsto x$.

6.5. *The quadratic transformations* $h_d : \Gamma_d \to \Gamma_{-d}$. Let g_d be the inverse of f_d. Define $h_d = g_d \circ f_d$.

$$x \mapsto x.$$

For $d = 1$, we have a map from Darboux to Thomson. In this case, a simple construction of h_1 is given by: Let P be a point on Darboux and P_i the perpendicular projections of P on the sides of ABC, let A^-, B^-, C^- be the midpoint of the altitudes of ABC, then $Q = h_1(P)$ is the intersection of the lines P_1A^-, P_2B^- and P_3C^-.

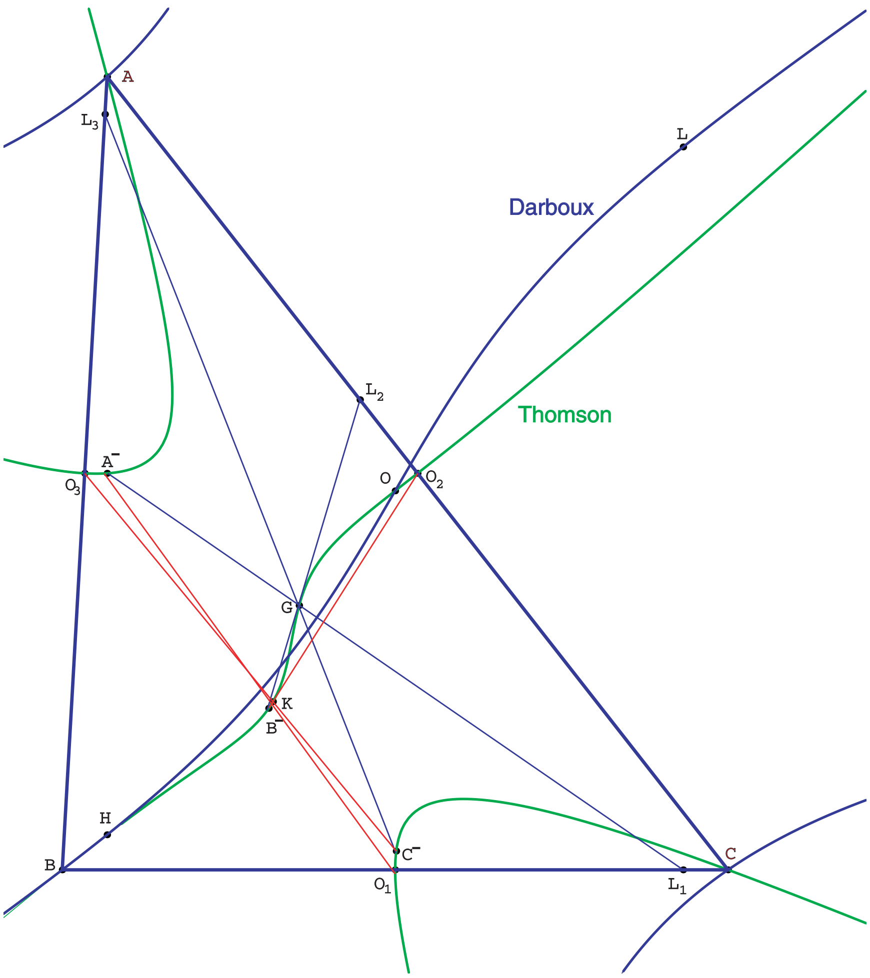

Figure 4. $G = h_1(L)$ and $K = h_1(O)$, h_1 : Darboux $\to$ Thomson

6.6. *Cevian, precevian, pedal and prepedal quadratic transformations.* 1. The Lucas cubic is the set of points P such that the cevian triangle of P is the pedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is $g_1 : x \mapsto x$.

2. The Lucas cubic is the set of points P such that the cevian triangle of P is the prepedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the isogonal of $g_1 : x \mapsto 6 - x$.

3. The Thomson cubic is the set of points P such that the precevian triangle of P is the pedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the inverse of $h_1 : x \mapsto x$.

4. The Thomson cubic is the set of points P such that the precevian triangle of P is the prepedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the symmetric of the inverse of $h_1 : x \mapsto 4 - x$.

This last transformation commutes with isogonality:

Proof: $x \mapsto 4-x \mapsto 6-(4-x) = 2+x$ and $x \mapsto 2-x \mapsto 4-(2-x) = 2+x$.

6.7. *Symmetry of center O of the Darboux cubic and induced transformations on Thomson and Lucas.* The symmetry is a linear transformation of the Darboux cubic: $x \mapsto 4 - x$. It induces via f_d and f_{-d} a quadratic involution of the Thomson cubic: $x \mapsto 4 - x$. And, via f_d and g_d, a quadratic involution of the Lucas cubic: $x \mapsto 4 - x$.

6.8. *Cyclocevian transformation.* The cyclocevian transformation [12] is an involution of the Lucas cubic. It is the symmetry relative to the line 3 of the group table: $x \mapsto 6 - x$.

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