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Brahmagupta Quadrilaterals

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Abstract. The Indian mathematician Brahmagupta made valuable contributions to mathematics and astronomy. He used Pythagorean triangles to construct general Heron triangles and cyclic quadrilaterals having integer sides, diagonals, and area, *i.e.*, Brahmagupta quadrilaterals. In this paper we describe a new numerical construction to generate an infinite family of Brahmagupta quadrilaterals from a Heron triangle.

1. Introduction

A triangle with integer sides and area is called a Heron triangle. If some of these elements are rationals that are not integers then we call it a rational Heron triangle. More generally, a polygon with integer sides, diagonals and area is called a Heron polygon. A rational Heron polygon is analogous to a rational Heron triangle. Brahmagupta's work on Heron triangles and cyclic quadrilaterals intrigued later mathematicians. This resulted in Kummer's complex construction to generate Heron quadrilaterals outlined in [2]. By a Brahmagupta quadrilateral we mean a cyclic Heron quadrilateral. In this paper we give a construction of Brahmagupta quadrilaterals from rational Heron triangles.

We begin with some well known results from circle geometry and trigonometry for later use.

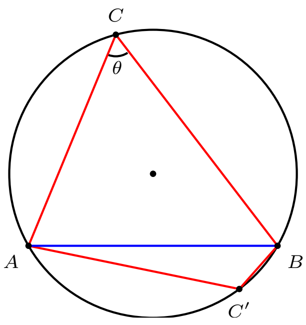

Figure 1

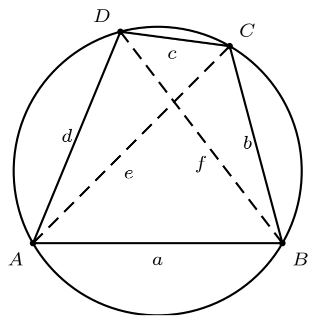

Figure 2

Figure 1 shows a chord AB of a circle of radius R. Let C and C' be points of the circle on opposite sides of AB. Then,

$$\angle ACB + \angle AC'B = \pi;$$
$$AB = 2R\sin\theta. \tag{1}$$

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Throughout our discussion on Brahmagupta quadrilaterals the following notation remains standard. $ABCD$ is a cyclic quadrilateral with vertices located on a circle in an order. $AB = a$, $BC = b$, $CD = c$, $DA = d$ represent the sides or their lengths. Likewise, $AC = e$, $BD = f$ represent the diagonals. The symbol $\triangle$ represents the area of $ABCD$. Brahmagupta's famous results are

$$e = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \tag{2}$$

$$f = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}, \tag{3}$$

$$\triangle = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{4}$$

where $s = \frac{1}{2}(a+b+c+d)$.

We observe that $d = 0$ reduces to Heron's famous formula for the area of triangle in terms of a, b, c. In fact the reader may derive Brahmagupta's expressions in (2), (3), (4) independently and see that they give two characterizations of a cyclic quadrilateral. We also observe that Ptolemy's theorem, viz., *the product of the diagonals of a cyclic quadrilateral equals the sum of the products of the two pairs of opposite sides*, follows from these expressions. In the next section, we give a construction of Brahmagupta quadrilaterals in terms of Heron angles. A Heron angle is one with rational sine and cosine. See [4]. Since

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2},$$

for $t = \tan\frac{\theta}{2}$, the angle θ is Heron if and only $\tan\frac{\theta}{2}$ is rational. Clearly, sums and differences of Heron angles are Heron angles. If we write, for triangle ABC, $t_1 = \tan\frac{A}{2}$, $t_2 = \tan\frac{B}{2}$, and $t_3 = \tan\frac{C}{2}$, then

$$a : b : c = t_1(t_2+t_3) : t_2(t_3+t_1) : t_3(t_1+t_2).$$

It follows that a triangle is rational if and only if its angles are Heron.

2. Construction of Brahmagupta quadrilaterals

Since the opposite angles of a cyclic quadrilateral are supplementary, we can always label the vertices of one such quadrilateral $ABCD$ so that the angles $A,\ B \le \frac{\pi}{2}$ and $C, D \ge \frac{\pi}{2}$. The cyclic quadrilateral $ABCD$ is a rectangle if and only if $A = B = \frac{\pi}{2}$; it is a trapezoid if and only if $A = B$. Let $\angle CAD = \angle CBD = \theta$. The cyclic quadrilateral $ABCD$ is rational if and only if the angles A, B and θ are Heron angles.

If $ABCD$ is a Brahmagupta quadrilateral whose sides AD and BC are not parallel, let E denote their intersection. [1] In Figure 3, let $EC = \alpha$ and $ED = \beta$. The triangles EAB and ECD are similar so that $\dfrac{AB}{CD} = \dfrac{EB}{ED} = \dfrac{EA}{EC} = \lambda$, say.

[1] Under the assumption that $A,\ B \le \frac{\pi}{2}$, these lines are parallel only if the quadrilateral is a rectangle.

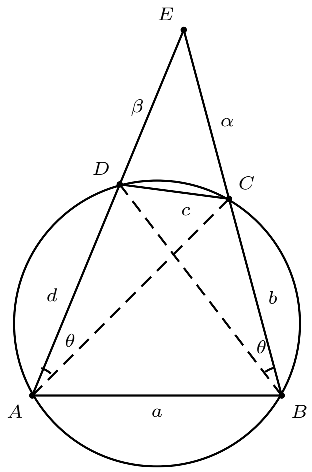

Figure 3

That is,

$$\frac{a}{c} = \frac{\alpha + b}{\beta} = \frac{\beta + d}{\alpha} = \lambda,$$

or

$$a = \lambda c, \quad b = \lambda\beta - \alpha, \quad d = \lambda\alpha - \beta, \qquad \lambda > \max\left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right). \tag{5}$$

Furthermore, from the law of sines, we have

$$e = 2R\sin B = 2R\sin D = \frac{R}{\rho}\cdot\alpha, \quad f = 2R\sin A = 2R\sin C = \frac{R}{\rho}\cdot\beta. \tag{6}$$

where ρ is the circumradius of triangle ECD. Ptolemy's theorem gives $ac + bd = ef$, and

$$\frac{R^2}{\rho^2}\cdot\alpha\beta = c^2\lambda + (\beta\lambda - \alpha)(\alpha\lambda - \beta)$$

This equation can be rewritten as

$$\begin{aligned}\left(\frac{R}{\rho}\right)^2 =& \lambda^2 - \frac{\alpha^2 + \beta^2 - c^2}{\alpha\beta}\lambda + 1\\ =& \lambda^2 - 2\lambda\cos E + 1\\ =& (\lambda - \cos E)^2 + \sin^2 E,\end{aligned}$$

or

$$\left(\frac{R}{\rho} - \lambda + \cos E\right)\left(\frac{R}{\rho} + \lambda - \cos E\right) = \sin^2 E.$$

Note that $\sin E$ and $\cos E$ are rational since E is a Heron angle. In order to obtain rational values for R and λ we put

$$\frac{R}{\rho} - \lambda - \cos E = t \sin E,$$
$$\frac{R}{\rho} + \lambda + \cos E = \frac{\sin E}{t},$$

for a rational number t. From these, we have

$$R = \frac{\rho}{2} \sin E \left(t + \frac{1}{t}\right) = \frac{c}{4}\left(t + \frac{1}{t}\right),$$
$$\lambda = \frac{1}{2} \sin E \left(\frac{1}{t} - t\right) - \cos E.$$

From the expression for R, it is clear that $t = \tan \frac{\theta}{2}$. If we set

$$t_1 = \tan \frac{D}{2} \quad \text{and} \quad t_2 = \tan \frac{C}{2}$$

for the Heron angles C and D, then

$$\cos E = \frac{(t_1 + t_2)^2 - (1 - t_1 t_2)^2}{(1 + t_1^2)(1 + t_2^2)}$$

and

$$\sin E = \frac{2(t_1 + t_2)(1 - t_1 t_2)}{(1 + t_1^2)(1 + t_2^2)}.$$

By choosing $c = t(1 + t_1^2)(1 + t_2^2)$, we obtain from (6)

$$\alpha = \frac{t t_1 (1 + t_1^2)(1 + t_2^2)^2}{(t_1 + t_2)(1 - t_1 t_2)}, \quad \beta = \frac{t t_2 (1 + t_1^2)^2 (1 + t_2^2)}{(t_1 + t_2)(1 - t_1 t_2)},$$

and from (5) the following simple rational parametrization of the sides and diagonals of the cyclic quadrilateral:

$$a = (t(t_1 + t_2) + (1 - t_1 t_2))(t_1 + t_2 - t(1 - t_1 t_2)),$$
$$b = (1 + t_1^2)(t_2 - t)(1 + t t_2),$$
$$c = t(1 + t_1^2)(1 + t_2^2),$$
$$d = (1 + t_2^2)(t_1 - t)(1 + t t_1),$$
$$e = t_1(1 + t^2)(1 + t_2^2),$$
$$f = t_2(1 + t^2)(1 + t_1^2).$$

This has area

$$\triangle = t_1 t_2 (2t(1 - t_1 t_2) - (t_1 + t_2)(1 - t^2))(2(t_1 + t_2)t + (1 - t_1 t_2)(1 - t^2)),$$

and is inscribed in a circle of diameter

$$2R = \frac{(1 + t_1^2)(1 + t_2^2)(1 + t^2)}{2}.$$

Replacing $t_1 = \frac{n}{m}$, $t_2 = \frac{q}{p}$, and $t = \frac{v}{u}$ for integers m, n, p, q, u, v in these expressions, and clearing denominators in the sides and diagonals, we obtain Brahmagupta quadrilaterals. Every Brahmagupta quadrilateral arises in this way.

3. Examples

Example 1. By choosing $t_1 = t_2 = \frac{n}{m}$ and putting $t = \frac{v}{u}$, we obtain a generic Brahmagupta trapezoid:

$$\begin{aligned} a =&(m^2u - n^2u + 2mnv)(2mnu - m^2v + n^2v), \\ b = d =&(m^2 + n^2)(nu - mv)(mu + nv), \\ c =&(m^2 + n^2)^2 uv, \\ e = f =&mn(m^2 + n^2)(u^2 + v^2), \end{aligned}$$

This has area

$$\triangle = 2m^2n^2(nu - mv)(mu + nv)((m+n)u - (m-n)v)((m+n)v - (m-n)u),$$

and is inscribed in a circle of diameter

$$2R = \frac{(m^2 + n^2)^2(u^2 + v^2)}{2}.$$

The following Brahmagupta trapezoids are obtained from simple values of t_1 and t, and clearing common divisors.

t_1	t	a	$b = d$	c	$e = f$	$\triangle$	$2R$
$1/2$	$1/7$	25	15	7	20	192	25
$1/2$	$2/9$	21	10	9	17	120	41
$1/3$	$3/14$	52	15	28	41	360	197
$1/3$	$3/19$	51	20	19	37	420	181
$2/3$	$1/8$	14	13	4	15	108	$65/4$
$2/3$	$3/11$	21	13	11	20	192	61
$2/3$	$9/20$	40	13	30	37	420	$1203/4$
$3/4$	$2/11$	25	25	11	30	432	61
$3/4$	$1/18$	17	25	3	26	240	$325/12$
$3/5$	$2/9$	28	17	12	25	300	$164/3$

Example 2. Let ECD be the rational Heron triangle with $c : \alpha : \beta = 14 : 15 : 13$. Here, $t_1 = \frac{2}{3}$, $t_2 = \frac{1}{2}$ (and $t_3 = \frac{4}{7}$). By putting $t = \frac{v}{u}$ and clearing denominators, we obtain Brahmagupta quadrilaterals with sides

$$a = (7u - 4v)(4u + 7v),\ b = 13(u - 2v)(2u + v),\ c = 65uv,\ d = 5(2u - 3v)(3u + 2v),$$

diagonals

$$e = 30(u^2 + v^2), \quad f = 26(u^2 + v^2),$$

and area

$$\triangle = 24(2u^2 + 7uv - 2v^2)(7u^2 - 8uv - 7v^2).$$

If we put $u = 3$, $v = 1$, we generate the particular one:

$$(a, b, c, d, e, f; \triangle) = (323, 91, 195, 165, 300, 260; 28416).$$

On the other hand, with $u = 11$, $v = 3$, we obtain a quadrilateral whose sides and diagonals are multiples of 65. Reduction by this factor leads to

$$(a, b, c, d, e, f; \triangle) = (65, 39, 33, 25, 52, 60; 1344).$$

This is inscribed in a circle of diameter 65. This latter Brahmagupta quadrilateral also appears in Example 4 below.

Example 3. If we take ECD to be a right triangle with sides $CD : EC : ED = m^2 + n^2 : 2mn : m^2 - n^2$, we obtain

$$\begin{aligned}
a =& (m^2 + n^2)(u^2 - v^2),\\
b =& ((m - n)u - (m + n)v)((m + n)u + (m - n)v),\\
c =& 2(m^2 + n^2)uv,\\
d =& 2(nu - mv)(mu + nv),\\
e =& 2mn(u^2 + v^2),\\
f =& (m^2 - n^2)(u^2 + v^2);\\
\triangle =& mn(m^2 - n^2)(u^2 + 2uv - v^2)(u^2 - 2uv - v^2).
\end{aligned}$$

Here, $\frac{u}{v} > \frac{m}{n}, \frac{m+n}{m-n}$. We give two very small Brahmagupta quadrilaterals from this construction.

n/m	v/u	a	b	c	d	e	f	$\triangle$	$2R$
1/2	1/4	75	13	40	36	68	51	966	85
1/2	1/5	60	16	25	33	52	39	714	65

Example 4. If the angle θ is chosen such that $A + B - \theta = \frac{\pi}{2}$, then the side BC is a diameter of the circumcircle of $ABCD$. In this case,

$$t = \tan\frac{\theta}{2} = \frac{1 - t_3}{1 + t_3} = \frac{t_1 + t_2 - 1 + t_1 t_2}{t_1 + t_2 + 1 - t_1 t_2}.$$

Putting $t_1 = \frac{n}{m}$, $t_2 = \frac{q}{p}$, and $t = \frac{(m+n)q-(m-n)p}{(m+n)p-(m-n)q}$, we obtain the following Brahmagupta quadrilaterals.

$$\begin{aligned}
a =& (m^2 + n^2)(p^2 + q^2),\\
b =& (m^2 - n^2)(p^2 + q^2),\\
c =& ((m + n)p - (m - n)q)((m + n)q - (m - n)p),\\
d =& (m^2 + n^2)(p^2 - q^2),\\
e =& 2mn(p^2 + q^2),\\
f =& 2pq(m^2 + n^2).
\end{aligned}$$

Here are some examples with relatively small sides.

t_1	t_2	t	a	b	c	d	e	f	$\triangle$
$2/3$	$1/2$	$3/11$	65	25	33	39	60	52	1344
$3/4$	$1/2$	$1/3$	25	7	15	15	24	20	192
$3/4$	$1/3$	$2/11$	125	35	44	100	120	75	4212
$6/7$	$1/3$	$1/4$	85	13	40	68	84	51	1890
$7/9$	$1/3$	$1/5$	65	16	25	52	63	39	1134
$8/9$	$1/2$	$3/7$	145	17	105	87	144	116	5760
$7/11$	$1/2$	$1/4$	85	36	40	51	77	68	2310
$8/11$	$1/3$	$1/6$	185	57	60	148	176	111	9240
$11/13$	$1/2$	$2/5$	145	24	100	87	143	116	6006

References

[1] J. R. Carlson, Determination of Heronian triangles, *Fibonnaci Quarterly*, 8 (1970) 499 – 506, 551.

[2] L. E. Dickson, *History of the Theory of Numbers*, vol. II, Chelsea, New York, New York, 1971; pp.171 – 201.

[3] C. Pritchard, Brahmagupta, *Math. Spectrum*, 28 (1995–96) 49–51.

[4] K. R. S. Sastry, Heron angles, *Math. Comput. Ed.*, 35 (2001) 51 – 60.

[5] K. R. S. Sastry, Heron triangles: a Gergonne cevian and median perspective, *Forum Geom.*, 1 (2001) 25 – 32.

[6] K. R. S. Sastry, Polygonal area in the manner of Brahmagupta, *Math. Comput. Ed.*, 35 (2001) 147–151.

[7] D. Singmaster, Some corrections to Carlson's "Determination of Heronian triangles", *Fibonnaci Quarterly*, 11 (1973) 157 – 158.

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