

## On the Fermat Lines

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**Abstract.** We study the triangle formed by three points each on a Fermat line of a given triangle, and at equal distances from the vertices. For two specific values of the common distance, the triangle degenerates into a line. The two resulting lines are the axes of the Steiner ellipse of the triangle.

### 1. The Fermat lines

This paper is on a variation of the theme of Bottema [2]. Bottema studied the triangles formed by three points each on an *altitude* of a given triangle, at equal distances from the respective vertices. See Figure 1. He obtained many interesting properties of this configuration. For example, these three points are collinear when the common distance is  $R \pm d$ , where  $R$  is the circumradius and  $d$  the distance between the circumcenter and the incenter of the reference triangle. The two lines containing the two sets of collinear points are perpendicular to each other at the incenter, and are parallel to the asymptotes of the Feuerbach hyperbola, the rectangular hyperbola through the vertices, the orthocenter, and the incenter. See Figure 2.

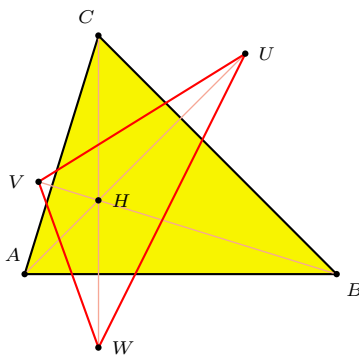


Figure 1

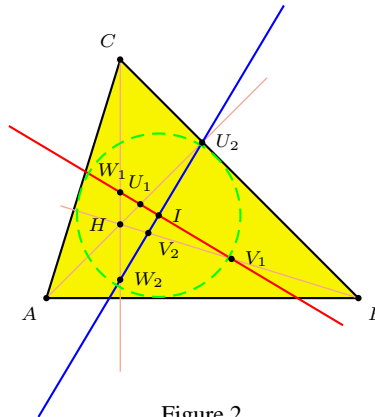


Figure 2

In this paper we consider the *Fermat lines*, which are the lines joining a vertex of the given triangle  $ABC$  to the apex of an equilateral triangle constructed on its opposite side. We label these triangles  $BCA_\epsilon$ ,  $CAB_\epsilon$ , and  $ABC_\epsilon$ , with  $\epsilon = +1$

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Publication Date: March 10, 2003. Guest Editor: Dick Klingens.

This paper is an extended revision of its Dutch version, *Over de lijnen van Fermat*, *Euclides*, 77 (2002) nr. 4, 188–193. This issue of *Euclides* is a tribute to O. Bottema (1900 – 1992). The author thanks Floor van Lamoen for translation into Dutch, and the editors of *Euclides* for permission to publish the present English version.

for those erected externally, and  $\epsilon = -1$  otherwise. There are 6 of such lines,  $AA_+$ ,  $BB_+$ ,  $CC_+$ ,  $AA_-$ ,  $BB_-$ , and  $CC_-$ . See Figure 3. The reason for choosing these lines is that, for  $\epsilon = \pm 1$ , the three segments  $AA_\epsilon$ ,  $BB_\epsilon$ , and  $CC_\epsilon$  have equal lengths  $\tau_\epsilon$  given by

$$\tau_\epsilon^2 = \frac{1}{2}(a^2 + b^2 + c^2) + \epsilon \cdot 2\sqrt{3}\Delta,$$

where  $a$ ,  $b$ ,  $c$  are the side lengths, and  $\Delta$  the area of triangle  $ABC$ . See, for example, [1, XXVII.3].

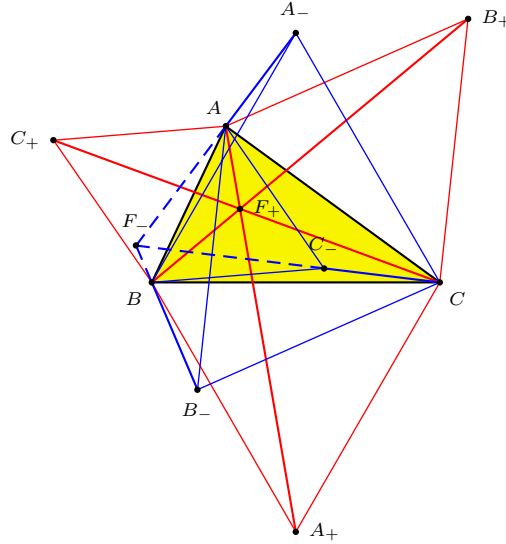


Figure 3

It is well known that the three Fermat lines  $AA_\epsilon$ ,  $BB_\epsilon$ , and  $CC_\epsilon$  intersect each other at the  $\epsilon$ -Fermat point  $F_\epsilon$  at  $60^\circ$  angles. The centers of the equilateral triangles  $BCA_\epsilon$ ,  $CAB_\epsilon$ , and  $ABC_\epsilon$  form the  $\epsilon$ -Napoleon equilateral triangle. The circumcircle of the  $\epsilon$ -Napoleon triangle has radius  $\frac{\tau_\epsilon}{3}$  and passes through the  $(-\epsilon)$ -Fermat point. See, for example, [5].

## 2. The triangles $\mathcal{T}_\epsilon(t)$

We shall label points on the Fermat lines by their distances from the corresponding vertices of  $ABC$ , positive in the direction from the vertex to the Fermat point, negative otherwise. Thus,  $A_+(t)$  is the unique point  $X$  on the positive Fermat line  $AF_+$  such that  $AX = t$ . In particular,

$$A_\epsilon(\tau_\epsilon) = A_\epsilon, \quad B_\epsilon(\tau_\epsilon) = B_\epsilon, \quad C_\epsilon(\tau_\epsilon) = C_\epsilon.$$

We are mainly interested in the triangles  $\mathcal{T}_\epsilon(t)$  whose vertices are  $A_\epsilon(t)$ ,  $B_\epsilon(t)$ ,  $C_\epsilon(t)$ , for various values of  $t$ . Here are some simple observations.

(1) The centroid of  $AA_+A_-$  is  $G$ . This is because the segments  $A_+A_-$  and  $BC$  have the same midpoint.

(2) The centers of the equilateral triangles  $BCA_+$  and  $BCA_-$  trisect the segment  $A_+A_-$ . Therefore, the segment joining  $A_\epsilon(\frac{\tau_\epsilon}{3})$  to the center of  $BCA_{-\epsilon}$  is parallel to the Fermat line  $AA_{-\epsilon}$  and has midpoint  $G$ .

(3) This means that  $A_\epsilon(\frac{\tau_\epsilon}{3})$  is the reflection of the  $A$ -vertex of the  $(-\epsilon)$ -Napoleon triangle in the centroid  $G$ . See Figure 4, in which we label  $A_+(\frac{\tau_+}{3})$  by  $X$  and  $A_-(\frac{\tau_-}{3})$  by  $X'$  respectively.

This is the same for the other two points  $B_\epsilon(\frac{\tau_\epsilon}{3})$  and  $C_\epsilon(\frac{\tau_\epsilon}{3})$ .

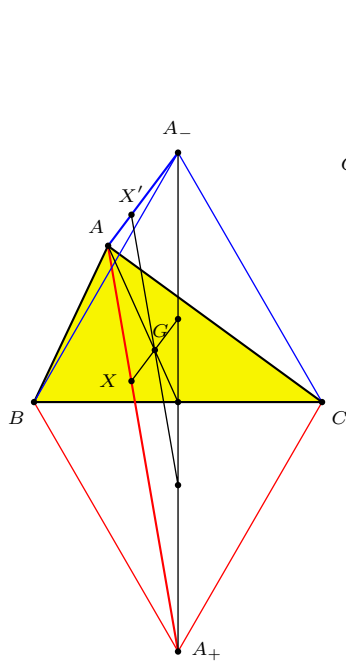


Figure 4

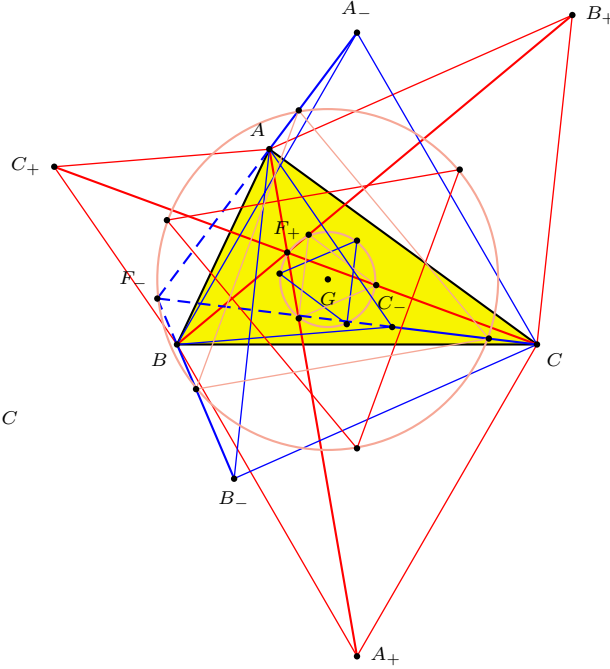


Figure 5

(4) It follows that the triangle  $\mathcal{T}_\epsilon(\frac{\tau_\epsilon}{3})$  is the reflection of the  $(-\epsilon)$ -Napoleon triangle in  $G$ , and is therefore equilateral.

(5) The circle through the vertices of  $\mathcal{T}_\epsilon(\frac{\tau_\epsilon}{3})$  and the  $(-\epsilon)$ -Napoleon triangle has radius  $\frac{\tau_{-\epsilon}}{3}$  and also passes through the Fermat point  $F_\epsilon$ . See Figure 5.

Since  $GA_\epsilon(\frac{\tau_\epsilon}{3}) = \frac{\tau_{-\epsilon}}{3}$ , (see Figure 4), the circle, center  $X$ , radius  $\frac{\tau_{-\epsilon}}{3}$ , passes through  $G$ . See Figure 6A. Likewise, the circle, center  $X'$ , radius  $\frac{\tau_\epsilon}{3}$  also passes through  $G$ . See Figure 6B. In these figures, we label

$$Y = A_+ \left( \frac{\tau_+ - \tau_-}{3} \right), \quad Z = A_+ \left( \frac{\tau_+ + \tau_-}{3} \right),$$

$$Y' = A_- \left( \frac{\tau_- - \tau_+}{3} \right), \quad Z' = A_- \left( \frac{\tau_+ + \tau_-}{3} \right).$$

It follows that  $GY$  and  $GZ$  are perpendicular to each other; so are  $GY'$  and  $GZ'$ .

(6) For  $\epsilon = \pm 1$ , the lines joining the centroid  $G$  to  $A_\epsilon(\frac{\tau_\epsilon + \tau_{-\epsilon}}{3})$  and  $A_\epsilon(\frac{\tau_\epsilon - \tau_{-\epsilon}}{3})$  are perpendicular to each other. Similarly, the lines joining  $G$  to  $B_\epsilon(\frac{\tau_\epsilon + \tau_{-\epsilon}}{3})$  and  $B_\epsilon(\frac{\tau_\epsilon - \tau_{-\epsilon}}{3})$  are perpendicular to each other; so are the lines joining  $G$  to  $C_\epsilon(\frac{\tau_\epsilon + \tau_{-\epsilon}}{3})$  and  $C_\epsilon(\frac{\tau_\epsilon - \tau_{-\epsilon}}{3})$ .

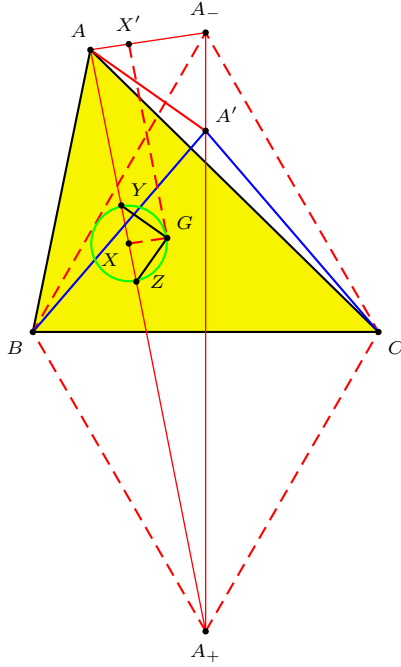


Figure 6A

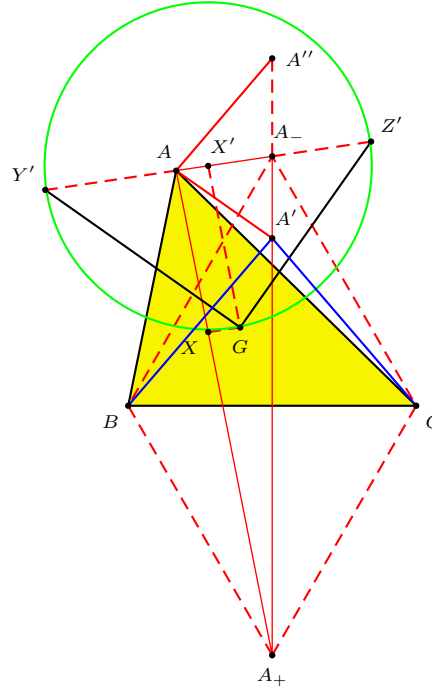


Figure 6B

In Figure 6A, since  $\angle XGY = \angle XYG$  and  $AXGX'$  is a parallelogram, the line  $GY$  is the bisector of angle  $XGX'$ , and is parallel to the bisector of angle  $A_+ A A_-$ . If the internal bisector of angle  $A_+ A A_-$  intersects  $A_+ A_-$  at  $A'$ , then it is easy to see that  $A'$  is the apex of the isosceles triangle constructed *inwardly* on  $BC$  with base angle  $\varphi$  satisfying

$$\cot \varphi = \frac{\tau_+ + \tau_-}{\sqrt{3}(\tau_+ - \tau_-)}. \tag{†}$$

Similarly, in Figure 6B, the line  $GZ'$  is parallel to the external bisector of the same angle. We summarize these as follows.

(7) The lines joining  $A_+(\frac{\tau_+ - \tau_-}{3})$  to  $A_-(\frac{\tau_- - \tau_+}{3})$  and  $A_+(\frac{\tau_+ + \tau_-}{3})$  to  $A_-(\frac{\tau_+ + \tau_-}{3})$  are perpendicular at  $G$ , and are respectively parallel to the internal and external bisectors of angle  $A_+ A A_-$ . Similarly, the two lines joining  $B_+(\frac{\tau_+ - \tau_-}{3})$  to  $B_-(\frac{\tau_- - \tau_+}{3})$  and  $B_+(\frac{\tau_+ + \tau_-}{3})$  to  $B_-(\frac{\tau_+ + \tau_-}{3})$  are perpendicular at  $G$ , being parallel to the internal and external bisectors of angle  $B_+ B B_-$ ; so are the lines joining

$C_+(\frac{\tau_+ - \tau_-}{3})$  to  $C_-(\frac{\tau_- - \tau_+}{3})$ , and  $C_+(\frac{\tau_+ + \tau_-}{3})$  to  $C_-(\frac{\tau_+ + \tau_-}{3})$ , being parallel to the internal and external bisectors of angle  $C_+CC_-$ .

### 3. Collinearity

What is interesting is that these 3 pairs of perpendicular lines in (7) above form the same right angles at the centroid  $G$ . Specifically, the six points

$$A_+(\frac{\tau_+ + \tau_-}{3}), B_+(\frac{\tau_+ + \tau_-}{3}), C_+(\frac{\tau_+ + \tau_-}{3}), A_-(\frac{\tau_+ + \tau_-}{3}), B_-(\frac{\tau_+ + \tau_-}{3}), C_-(\frac{\tau_+ + \tau_-}{3})$$

are collinear with the centroid  $G$  on a line  $\mathcal{L}_+$ ; so are the 6 points

$$A_+(\frac{\tau_+ - \tau_-}{3}), B_+(\frac{\tau_+ - \tau_-}{3}), C_+(\frac{\tau_+ - \tau_-}{3}), A_-(\frac{\tau_+ - \tau_-}{3}), B_-(\frac{\tau_+ - \tau_-}{3}), C_-(\frac{\tau_+ - \tau_-}{3})$$

on a line  $\mathcal{L}_-$  through  $G$ . See Figure 7. To justify this, we consider the triangle  $\mathcal{T}_\epsilon(t) := A_\epsilon(t)B_\epsilon(t)C_\epsilon(t)$  for varying  $t$ .

(8) For  $\epsilon = \pm 1$ , the triangle  $\mathcal{T}_\epsilon(t)$  degenerates into a line containing the centroid  $G$  if and only if  $t = \frac{\tau_\epsilon + \delta\tau_{-\epsilon}}{3}$ ,  $\delta = \pm 1$ .

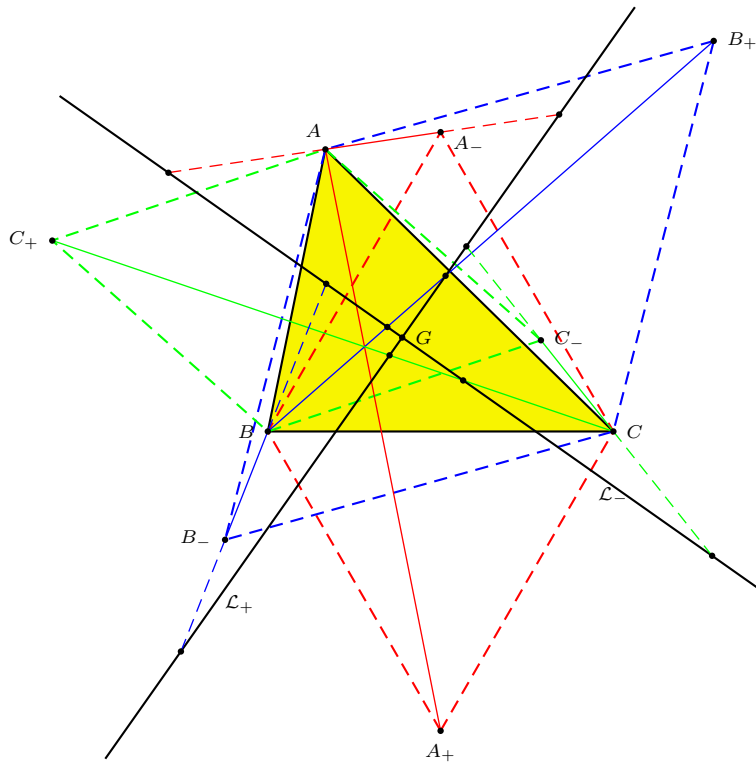


Figure 7

#### 4. Barycentric coordinates

To prove (8) and to obtain further interesting geometric results, we make use of coordinates. Bottema has advocated the use of homogeneous barycentric coordinates. See [3, 6]. Let  $P$  be a point in the plane of triangle  $ABC$ . With reference to  $ABC$ , the homogeneous barycentric coordinates of  $P$  are the ratios of signed areas

$$(\triangle PBC : \triangle PCA : \triangle PAB).$$

The coordinates of the vertex  $A_+$  of the equilateral triangle  $BCA_+$ , for example, are  $(-\frac{\sqrt{3}}{4}a^2 : \frac{1}{2}ab \sin(C + 60^\circ) : \frac{1}{2}ca \sin(B + 60^\circ))$ , which can be rewritten as

$$A_+ = (-2\sqrt{3}a^2 : \sqrt{3}(a^2 + b^2 - c^2) + 4\Delta : \sqrt{3}(c^2 + a^2 - b^2) + 4\Delta).$$

More generally, for  $\epsilon = \pm 1$ , the vertices of the equilateral triangles erected on the sides of triangle  $ABC$  are the points

$$\begin{aligned} A_\epsilon &= (-2\sqrt{3}a^2 : \sqrt{3}(a^2 + b^2 - c^2) + 4\epsilon\Delta : \sqrt{3}(c^2 + a^2 - b^2) + 4\epsilon\Delta), \\ B_\epsilon &= (\sqrt{3}(a^2 + b^2 - c^2) + 4\epsilon\Delta : -2\sqrt{3}b^2 : \sqrt{3}(b^2 + c^2 - a^2) + 4\epsilon\Delta), \\ C_\epsilon &= (\sqrt{3}(c^2 + a^2 - b^2) + 4\epsilon\Delta : \sqrt{3}(b^2 + c^2 - a^2) + 4\epsilon\Delta : -2\sqrt{3}c^2). \end{aligned}$$

Note that in each case, the coordinate sum is  $8\epsilon\Delta$ . From this we easily compute the coordinates of the centroid by simply adding the corresponding coordinates of the three vertices.

(9A) For  $\epsilon = \pm 1$ , triangles  $A_\epsilon B_\epsilon C_\epsilon$  and  $ABC$  have the same centroid.

Sometimes it is convenient to work with *absolute* barycentric coordinates. For a finite point  $P = (u : v : w)$ , we obtain the absolute barycentric coordinates by normalizing its homogeneous barycentric coordinates, namely, by dividing by the coordinate sum. Thus,

$$P = \frac{1}{u + v + w}(uA + vB + wC),$$

provided  $u + v + w$  is nonzero.

The absolute barycentric coordinates of the point  $A_\epsilon(t)$  can be easily written down. For each value of  $t$ ,

$$A_\epsilon(t) = \frac{1}{\tau_\epsilon}((\tau_\epsilon - t)A + t \cdot A_\epsilon),$$

and similarly for  $B_\epsilon(t)$  and  $C_\epsilon(t)$ .

This, together with (9A), leads easily to the more general result.

(9B) For arbitrary  $t$ , the triangles  $\mathcal{T}_\epsilon(t)$  and  $ABC$  have the same centroid.

**5. Area of  $\mathcal{T}_\epsilon(t)$**

Let  $X = (x_1 : x_2 : x_3)$ ,  $Y = (y_1 : y_2 : y_3)$  and  $Z = (z_1 : z_2 : z_3)$  be finite points with homogeneous coordinates with respect to triangle  $ABC$ . The *signed* area of the oriented triangle  $XYZ$  is

$$\frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}}{(x_1 + x_2 + x_3)(y_1 + y_2 + y_3)(z_1 + z_2 + z_3)} \cdot \Delta.$$

A proof of this elegant formula can be found in [1, VII] or [3]. A direct application of this formula yields the area of triangle  $\mathcal{T}_\epsilon(t)$ .

(10) The area of triangle  $\mathcal{T}_\epsilon(t)$  is

$$\frac{3\sqrt{3}\epsilon}{4} \left( t - \frac{\tau_\epsilon + \tau_{-\epsilon}}{3} \right) \left( t - \frac{\tau_\epsilon - \tau_{-\epsilon}}{3} \right) \Delta.$$

Statement (8) follows immediately from this formula and (9B).

(11)  $\mathcal{T}_\epsilon(t)$  has the same area as  $ABC$  if and only if  $t = 0$  or  $\frac{2\tau_\epsilon}{3}$ . In fact, the two triangles  $\mathcal{T}_+(\frac{2\tau_+}{3})$  and  $\mathcal{T}_-(\frac{2\tau_-}{3})$  are symmetric with respect to the centroid. See Figures 8A and 8B.

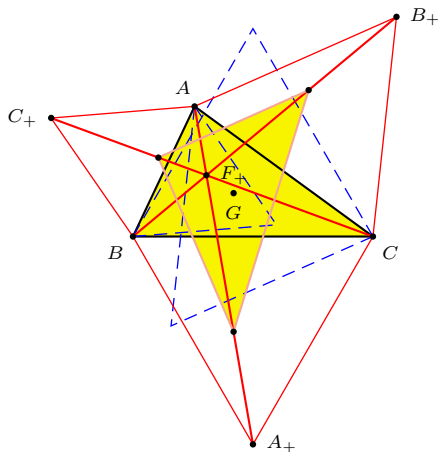


Figure 8A

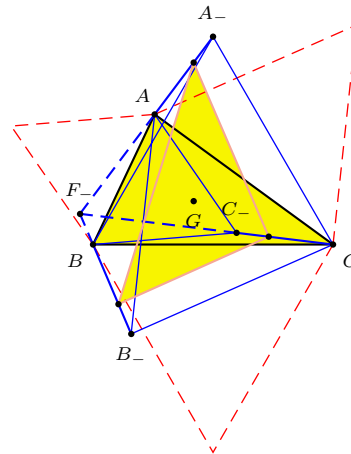


Figure 8B

**6. Kiepert hyperbola and Steiner ellipse**

The existence of the line  $\mathcal{L}_-$  (see §3) shows that the internal bisectors of the angles  $A_+AA_-$ ,  $B_+BB_-$ , and  $C_+CC_-$  are parallel. These bisectors contain the apexes  $A'$ ,  $B'$ ,  $C'$  of isosceles triangles constructed inwardly on the sides with the same base angle given by  $(\dagger)$ . It is well known that  $A'B'C'$  and  $ABC$  are perspective at a point on the Kiepert hyperbola, the rectangular circum-hyperbola

through the orthocenter and the centroid. This perspector is necessarily an infinite point (of an asymptote of the hyperbola). In other words, the line  $\mathcal{L}_-$  is parallel to an asymptote of this rectangular hyperbola.

(12) The lines  $\mathcal{L}_\pm$  are the parallels through  $G$  to the asymptotes of the Kiepert hyperbola.

(13) It is also known that the asymptotes of the Kiepert hyperbola are parallel to the axes of the Steiner in-ellipse, (see [4]), the ellipse that touches the sides of triangle  $ABC$  at their midpoints, with center at the centroid  $G$ . See Figure 9.

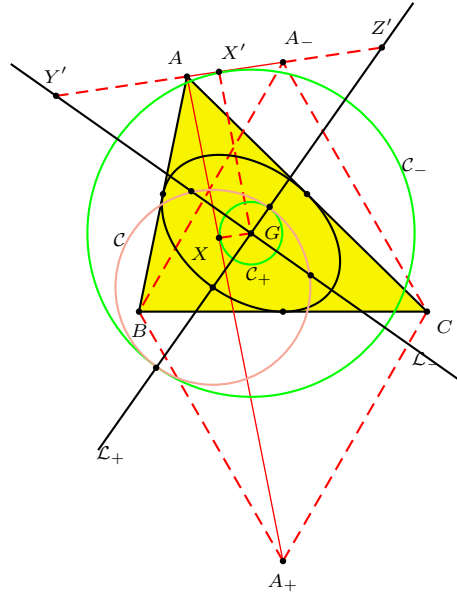


Figure 9

(14) The Steiner in-ellipse has major and minor axes of lengths  $\frac{\tau_+ \pm \tau_-}{3}$ . From this, we have the following construction of its foci. See Figure 9.

- Construct the concentric circles  $\mathcal{C}_\pm$  at  $G$  through  $A_\epsilon(\frac{\tau_\epsilon}{3})$ .
- Construct a circle  $\mathcal{C}$  with center on  $\mathcal{L}_+$  tangent to the circles  $\mathcal{C}_+$  internally and  $\mathcal{C}_-$  externally. There are two such circles; any one of them will do.
- The intersections of the circle  $\mathcal{C}$  with the line  $\mathcal{L}_-$  are the foci of Steiner in-ellipse.

We conclude by recording the homogeneous barycentric coordinates of the two foci of the Steiner in-ellipse. Let

$$Q = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2.$$



The line  $\mathcal{L}_-$  containing the two foci has infinite point

$$I_-^\infty = ((b-c)(a(a+b+c) - (b^2 + bc + c^2) - \sqrt{Q}), \\ (c-a)(b(a+b+c) - (c^2 + ca + a^2) - \sqrt{Q}), \\ (a-b)(c(a+b+c) - (a^2 + ab + b^2) - \sqrt{Q})).$$

As a vector, this has square length  $2\sqrt{Q}(f + g\sqrt{Q})$ , where

$$f = \sum_{\text{cyclic}} a^6 - bc(b^4 + c^4) + a^2bc(ab + ac - bc), \\ g = \sum_{\text{cyclic}} a^4 - bc(b^2 + c^2 - a^2).$$

Since the square distance from the centroid to each of the foci is  $\frac{1}{9}\sqrt{Q}$ , these two foci are the points

$$G \pm \frac{1}{3\sqrt{2}(f + g\sqrt{Q})} I_-^\infty.$$

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