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Similar Pedal and Cevian Triangles

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Abstract. The only point with similar pedal and cevian triangles, other than the orthocenter, is the isogonal conjugate of the Parry reflection point.

1. Introduction

We begin with notation. Let ABC be a triangle with sidelengths a, b, c, orthocenter H, and circumcenter O. Let K_A, K_B, K_C denote the vertices of the tangential triangle, O_A, O_B, O_C the reflections of O in A, B, C, and A_S, B_S, C_S the reflections of the vertices of A in BC, of B in CA, and of C in AB. Let

$M^* =$ isogonal conjugate of a point M;

$\overline{M} =$ inverse of M in the circumcircle;

$\measuredangle LL' =$ the measure, modulo π, of the directed angle of the lines L, L';

$S_A = bc\cos A = \frac{1}{2}(b^2+c^2-a^2)$, with S_B and S_C defined cyclically;

$x:y:z =$ barycentric coordinates relative to triangle ABC;

$\Gamma_A =$ circle with diameter K_AO_A, with circles Γ_B and Γ_C defined cyclically.

The circle Γ_A passes through the points B_S, C_S and is the locus of M such that $\measuredangle B_SMC_S = -2\measuredangle BAC$. An equation for Γ_A, in barycentrics, is

$$2S_A\left(a^2yz+b^2zx+c^2xy\right)+\left(b^2c^2x+2c^2S_Cy+2b^2S_Bz\right)(x+y+z)=0.$$

Consider a triangle $A'B'C'$, where A', B', C' lie respectively on the sidelines BC, CA, AB. The three circles $AB'C'$, $BC'A'$, $CA'B'$ meet in a point S called the Miquel point of $A'B'C'$. See [2, pp.131–135]. The point S (or $\overline{S}$) is the only point whose pedal triangle is directly (or indirectly) similar to $A'B'C'$.

The circles Γ_A, Γ_B, Γ_C have a common point T: the Parry reflection point, X_{399} in [3]; the three radical axes TA_S, TB_S, TC_S are the reflections with respect to a sideline of ABC of the parallel to the Euler line going through the opposite vertex. See [3, 4], and Figure 1. T lies on the circle $(O, 2R)$, on the Neuberg cubic, and is the antipode of O on the Stammler hyperbola. See [1].

2. Similar triangles

Let $A'B'C'$ be the cevian triangle of a point $P = p:q:r$.

Lemma 1. *The pedal and cevian triangles of P are directly (or indirectly) similar if and only if P (or $\overline{P}$) lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$.*

Proof. This is an immediate consequence of the properties of the Miquel point above. □

Lemma 2. *A, B', C', P are concyclic if and only if P lies on the circle BCH.*

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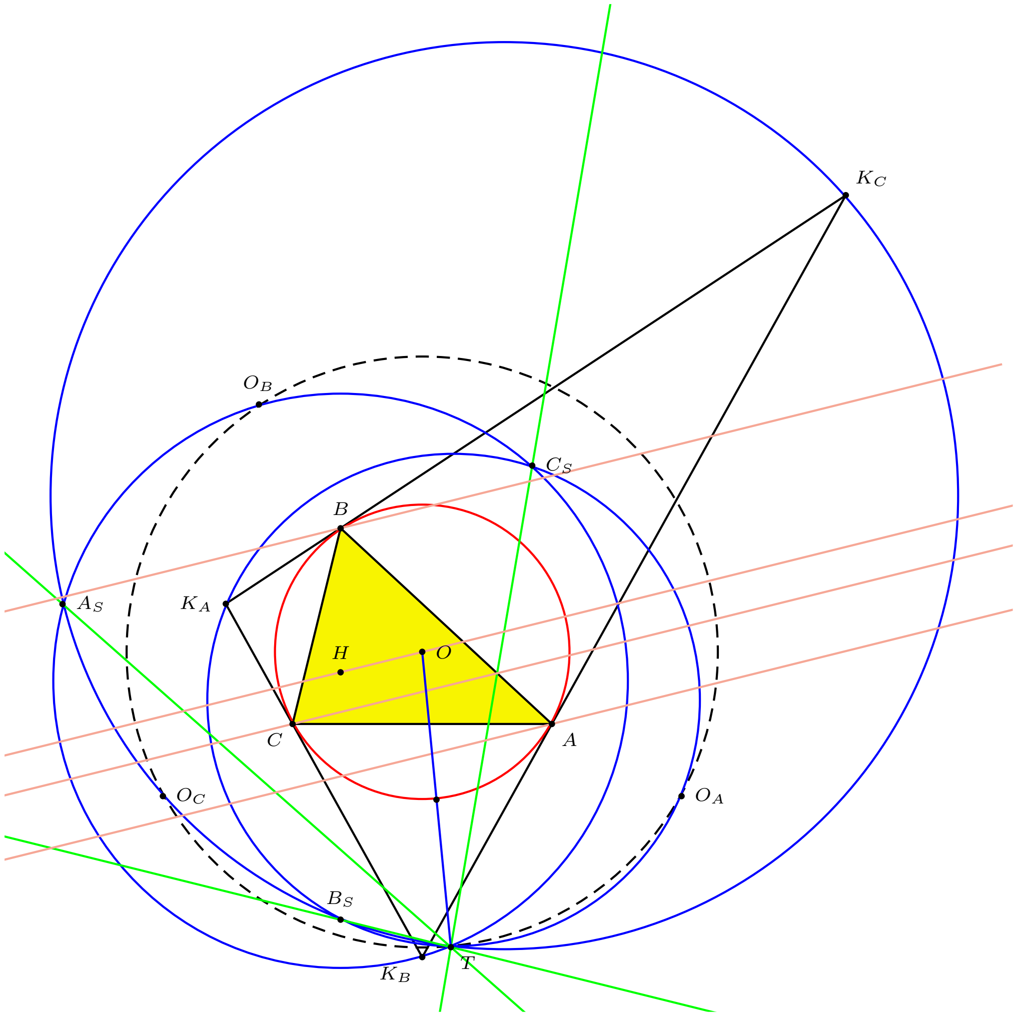

Figure 1

Proof. A, B', C' and P are concyclic $\Leftrightarrow \measuredangle B'PC' = \measuredangle B'AC' \Leftrightarrow \measuredangle BPC = \measuredangle BHC \Leftrightarrow P$ lies on the circle BCH. $\square$

Proposition 3. *The pedal and cevian triangles of P are directly similar only in the trivial case of $P = H$.*

Proof. By Lemma 1, the pedal and cevian triangles of P are directly similar if and only if P lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$. By Lemma 2, P lies on the three circles BCH, CAH, ABH. Hence, $P = H$. $\square$

Lemma 4. A, B', C', $\overline{P}$ *are concyclic if and only if* P^* *lies on the circle* Γ_A.

Proof. If $P = p : q : r$, the circle Φ_A passing through A, B', C' is given by

$$a^2yz + b^2zx + c^2xy - p(x+y+z)\left(\frac{c^2}{p+q}y + \frac{b^2}{p+r}z\right) = 0,$$

and its inverse in the circumcircle is the circle $\overline{\Phi}_A$ given by

$$\begin{aligned} &(a^2(p^2-qr) + (b^2-c^2)p(q-r))(a^2yz + b^2zx + c^2xy) \\ - \; & pa^2(x+y+z)(c^2(p+r)y + b^2(p+q)z) = 0. \end{aligned}$$

Since Φ_A contains $\overline{P}$, its inverse $\overline{\Phi}_A$ contains P. Changing (p, q, r) to (x, y, z) gives the locus of P satisfying $\overline{P} \in \Phi_A$. Then changing (x, y, z) to $\left(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}\right)$ gives the locus $\widehat{\overline{\Phi}}_A$ of the point P^* such that $\overline{P} \in \Phi_A$. By examination, $\widehat{\overline{\Phi}}_A = \Gamma_A$. □

Proposition 5. *The pedal and cevian triangles of P are indirectly similar if and only if P is the isogonal conjugate of the Parry reflection point.*

Proof. By Lemma 1, the pedal and cevian triangles of P are indirectly similar if and only if $\overline{P}$ lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$. By Lemma 4, P^* lies on each of the circles Γ_A, Γ_B, Γ_C. Hence, $P^* = T$, and $P = T^*$. □

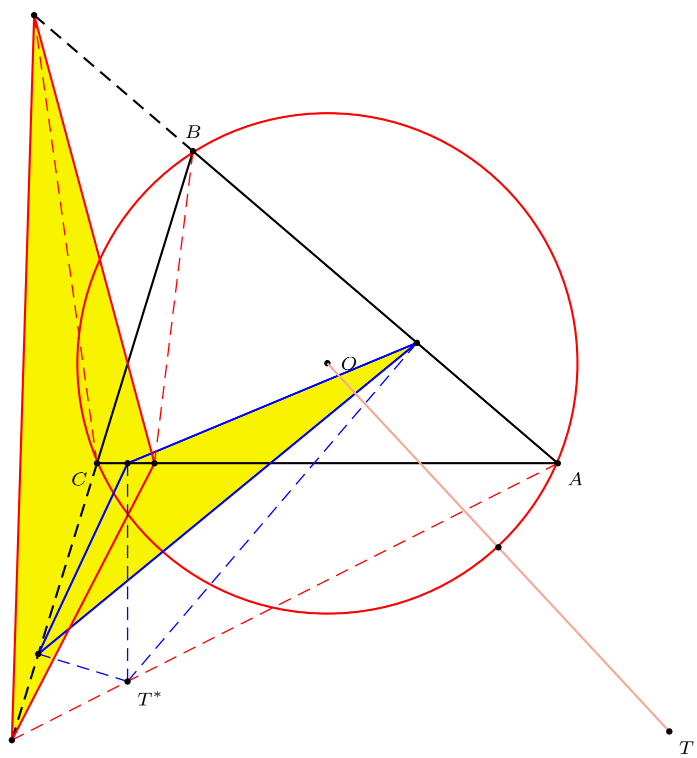

Figure 2

Remarks. (1) The isogonal conjugate of X_{399} is X_{1138} in [3]: this point lies on the Neuberg cubic.

(2) We can deduce Lemma 4 from the relation $\measuredangle B'\overline{P}C' - \measuredangle B_sP^*C_s = \measuredangle BAC$, which is true for every point P in the plane of ABC except the vertices A, B, C.

(3) As two indirectly similar triangles are orthologic and as the pedal and cevian triangles of P are orthologic if and only if P^* lies on the Stammler hyperbola, a point with indirectly similar cevian and pedal triangles must be the isogonal conjugate of a point of the Stammler hyperbola.

References

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