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A Tetrahedral Arrangement of Triangle Centers

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Abstract. We present a graphic scheme for indexing 25 collinearities of 17 triangle centers three at a time. The centers are used to label vertices and edges of nested polyhedra. Two new triangle centers are introduced to make this possible.

1. Introduction

Collinearities of triangle centers which are defined in apparently different ways has been of interest to geometers since it was first noticed that the orthocenter, centroid, and circumcenter are collinear, lying on Euler's line. Kimberling [3] lists a great many collinearites, including many more points on Euler's line. The object of this note is to present a three-dimensional graphical summary of 25 three-center collinearities involving 17 centers, in which the centers are represented as vertices and edge midpoints of nested polyhedra: a tetrahedron circumscribing an octahedron which then circumscribes a cubo-octahedron. Such a symmetric collection of collinearities may be a useful mnemonic. Probably the reason why this has not been recognized before is that two of the vertices of the tetrahedron represent previously undescribed centers. First we describe two new centers, which Kimberling lists as X_{1276} and X_{1277} in his *Encyclopedia of Triangle Centers* [3]. Then we describe the tetrahedron and work inward to the cubo-octahedron.

2. Perspectors and the excentral triangle

The excentral triangle, $\mathbf{T}_\mathrm{x}$, of a triangle $\mathbf{T}$ is the triangle whose vertices are the excenters of $\mathbf{T}$. Let $\mathbf{T}_+$ be the triangle whose vertices are the apices of equilateral triangles erected outward on the sides of $\mathbf{T}$. Similarly let $\mathbf{T}_-$ be the triangle whose vertices are the apices of equilateral triangles erected inward on the sides of $\mathbf{T}$. It happens that $\mathbf{T}_\mathrm{x}$ is in perspective from $\mathbf{T}_+$ from a point V_+, a previously undescribed triangle center now listed as X_{1276} in [3], and that $\mathbf{T}_\mathrm{x}$ is also in perspective from $\mathbf{T}_-$ from another new center V_- listed as X_{1277} in [3]. See Figure 1.

For $\varepsilon = \pm 1$, the homogeneous trilinear coordinates of V_ε are

$$1 - v_a + v_b + v_c : 1 + v_a - v_b + v_c : 1 + v_a + v_b - v_c,$$

where $v_a = -\frac{2}{\sqrt{3}} \sin(A + \varepsilon \cdot 60°)$ etc.

It is well known that $\mathbf{T}_\mathrm{x}$ and $\mathbf{T}$ are in perspective from the incenter I. Define $\mathbf{T}^*$ as the triangle whose vertices are the reflections of the vertices of $\mathbf{T}$ in the opposite sides. Then $\mathbf{T}_\mathrm{x}$ and $\mathbf{T}^*$ are in perspective from a point W listed as X_{484} in [3]. See Figure 2. The five triangles $\mathbf{T}$, $\mathbf{T}_\mathrm{x}$, $\mathbf{T}_+$, $\mathbf{T}_-$, and $\mathbf{T}^*$ are pairwise in perspective, giving 10 perspectors. Denote the perspector of two triangles by enclosing the two triangles in brackets, so, for example $[\mathbf{T}_\mathrm{x}, \mathbf{T}] = I$.

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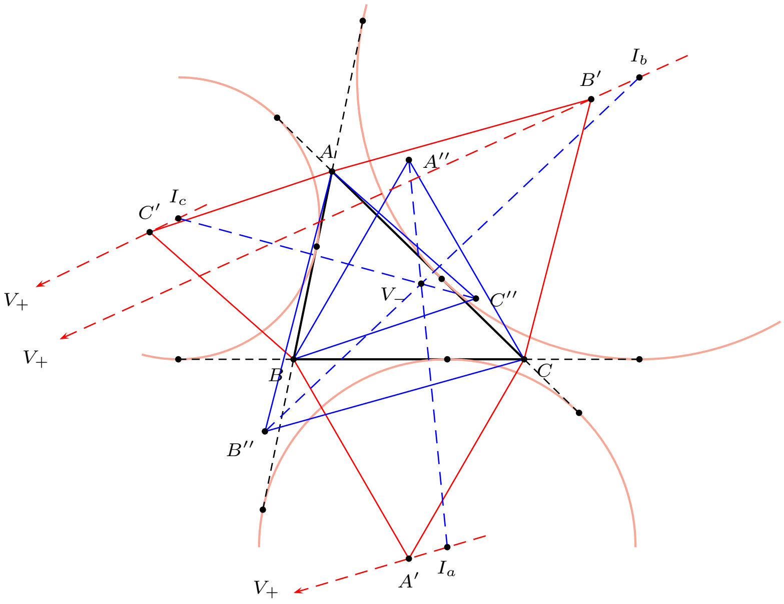

Figure 1

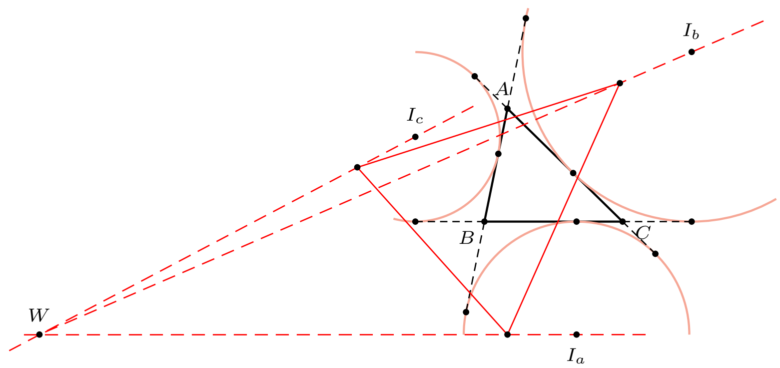

Figure 2

Here is a list of the 10 perspectors with their names and ETC numbers:

$[\mathbf{T}, \mathbf{T}_+]$	F_+	First Fermat point	X_{13}
$[\mathbf{T}, \mathbf{T}_-]$	F_-	Second Fermat point	X_{14}
$[\mathbf{T}, \mathbf{T}^*]$	H	Orthocenter	X_4
$[\mathbf{T}, \mathbf{T}_\mathrm{x}]$	I	Incenter	X_1
$[\mathbf{T}_+, \mathbf{T}_-]$	O	Circumcenter	X_3
$[\mathbf{T}_+, \mathbf{T}^*]$	J_-	Second isodynamic point	X_{16}
$[\mathbf{T}_-, \mathbf{T}^*]$	J_+	First isodynamic point	X_{15}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}^*]$	W	First Evans perspector	X_{484}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}_+]$	V_+	Second Evans perspector	X_{1276}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}_-]$	V_-	Third Evans perspector	X_{1277}

3. Collinearities among the ten perspectors

As in [2], we shall write $\mathcal{L}(X, Y, Z, \dots)$ to denote the line containing X, Y, Z, ... The following collinearities may be easily verified:

$$\begin{array}{lll} \mathcal{L}(I, O, W), & \mathcal{L}(I, J_-, V_-), & \mathcal{L}(I, J_+, V_+), \\ \mathcal{L}(V_+, H, V_-), & \mathcal{L}(W, F_+, V_-), & \mathcal{L}(W, F-, V_+). \end{array}$$

What is remarkable is that all five triangles are involved in each collinearity, with $\mathbf{T}_\mathrm{x}$ used twice. For example, rewrite $\mathcal{L}(I, O, W)$ as

$$\mathcal{L}([\mathbf{T}, \mathbf{T}_\mathrm{x}], [\mathbf{T}_+, \mathbf{T}_-], [\mathbf{T}_\mathrm{x}, \mathbf{T}^*])$$

to see this. The six collinearites have been stated so that the first and third perspectors involve $\mathbf{T}_\mathrm{x}$, with the perspector of the remaining two triangles listed second. This lends itself to a graphical representation as a tetrahedron with vertices labelled with I, V_+, V_-, and W, and the edges labelled with the perspectors collinear with the vertices. See Figure 3. When these centers are actually constructed, they may not be in the order listed in these collinearities. For example, O is not necessarily between I and W. There is another collinearity which we do not use, however, namely, $\mathcal{L}(O, J_+, J_-)$, which is the Brocard axis. Triangle $\mathbf{T}_\mathrm{x}$ is not involved in any of the perspectors in this collinearity.

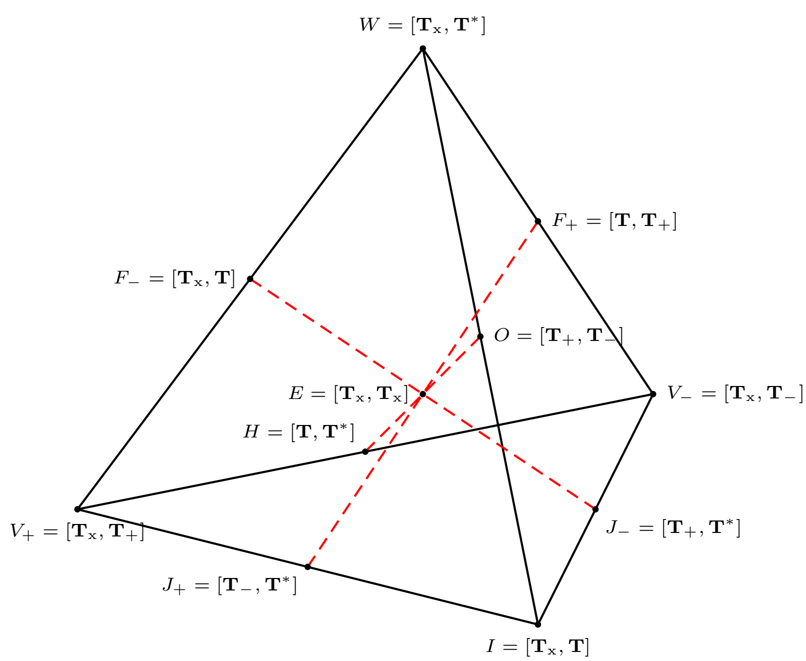

Figure 3

If we label each edge of the tetrahedron at its midpoint by the middle center listed in each of the collinearities above, then opposite edge midpoints are pairs of isogonal conjugates: H and O, J_+ and F_+, and J_- and F_-. Also the lines $\mathcal{L}(O, H)$, $\mathcal{L}(F_+, J_+)$, and $\mathcal{L}(F_-, J_-)$ are parallel to the Euler line, and may be interpreted as intersecting at the Euler infinity point E, listed as X_{30} in [3]. This adds three more collinearities to the tetrahedral scheme:

$$\mathcal{L}(O, E, H),\ \mathcal{L}(F_+, E, J_+),\ \mathcal{L}(F_-, E, J_-).$$

The five triangles $\mathbf{T}$, $\mathbf{T}_+$, $\mathbf{T}_-$, $\mathbf{T}^*$, and $\mathbf{T}_\mathrm{x}$ are all inscribed in Neuberg's cubic curve. Now consider a triangle $\mathbf{T}'_\mathrm{x}$ in perspective with $\mathbf{T}_\mathrm{x}$ and inscribed in the cubic with vertices very close to those of $\mathbf{T}_\mathrm{x}$ (the excenters of $\mathbf{T}$). The lines of perspective of $\mathbf{T}'_\mathrm{x}$ and $\mathbf{T}_\mathrm{x}$ approach the tangents to Neuberg's cubic at the vertices of $\mathbf{T}_\mathrm{x}$ as $\mathbf{T}'_\mathrm{x}$ approaches $\mathbf{T}_\mathrm{x}$. These tangents are known to be parallel to the Euler line and may be thought of as converging at the Euler point at infinity, $E = X_{30}$. So we can write $E = [\mathbf{T}_\mathrm{x}, \mathbf{T}_\mathrm{x}]$, interpreting this to mean that $\mathbf{T}_\mathrm{x}$ is in perspective from itself from E. I propose the term "ipseperspector" for such a point, from the Latin "ipse" for self. Note that the notion of ipseperspector is dependent on the curve circumscribing the triangle $\mathbf{T}$. A well-known example of an ipseperspector for a triangle curcumscribed in Neuberg's cubic is X_{74}, this being the point where the tangents to the curve at the vertices of $\mathbf{T}$ intersect.

4. Further nested polyhedra

We shall encounter other named centers, which are listed here for reference:

G	Centroid	X_2
K	Symmedian (Lemoine) point	X_6
N_+	First Napoleon point	X_{17}
N_-	Second Napoleon point	X_{18}
N_+^*	Isogonal conjugate of N_+	X_{61}
N_-^*	Isogonal conjugate of N_-	X_{62}

The six midpoints of the edges of the tetrahedron may be considered as the vertices of an inscribed octahedron. This leads to indexing more collinearities in the following way: label the midpoint of each edge of the octahedron by the point where the lines indexed by opposite edges meet. For example, opposite edges of the octahedron $\mathcal{L}(F_+, J_-)$ and $\mathcal{L}(F_-, J_+)$ meet at the centroid G. We can then write two 3-point collinearities as $\mathcal{L}(F_+, G, J_-)$ and $\mathcal{L}(F_-, G, J_+)$. Now the edges adjacent to both of these edges index the lines $\mathcal{L}(F_+, F_-)$ and $\mathcal{L}(J_+, J_-)$, which meet at the symmedian point K. This gives two more 3-point collinearities, $\mathcal{L}(F_+, K, F_-)$ and $\mathcal{L}(J_+, K, J_-)$. Note that G and K are isogonal conjugates. This pattern persists with the other pairs of opposite edges of the octahedron.

The intersections of other lines represented as opposite edges intersect at the Napoleon points and their isogonal conjugates. When we consider the four vertices O, F_-, H, and J_- of the octahedron, four more 3-point collinearities are indexed in the same manner: $\mathcal{L}(O, N_-^*, J_-)$, $\mathcal{L}(H, N_-^*, F_-)$, $\mathcal{L}(O, N_-, F_-)$, and $\mathcal{L}(H, N_-, J_-)$. Similarly, from vertices O, F_+ , H, and J_+, four more 3-point collinearites arise in the same indexing process: $\mathcal{L}(O, N_+^*, J_+)$, $\mathcal{L}(H, N_+^*, F_+)$, $\mathcal{L}(O, N_+, F_+)$, and $\mathcal{L}(H, N_+, J_+)$. So each of the twelve edges of the octahedron indexes a different 3-point collinearity.

Let us carry this indexing scheme further. Now consider the midpoints of the edges of the octahedron to be the vertices of a polyhedron inscribed in the octahedron. This third nested polyhedron is a cubo-octahedron: it has eight triangular faces, each of which is coplanar with a face of the octahedron, and six square faces. Yet again more 3-point collinearities are indexed, but this time by the triangular faces of the cubo-octahedron. It happens that the three vertices of each triangular face of the cubo-octahedron, which inherit their labels as edges of the octahedron, are collinear in the plane of the basic triangle $\mathbf{T}$. Opposite edges of the octahedron have the same point labelling their midpoints, so opposite triangular faces of the cubo-octahedron are labelled by the same three centers. This means that there are four instead of eight collinearities indexed by the triangular faces: $\mathcal{L}(G, N_+, N_-^*)$,$\mathcal{L}(G, N_-, N_+^*)$, $\mathcal{L}(K, N_+, N_-)$, and $\mathcal{L}(K, N_-^*, N_+^*)$. See Figure 4.

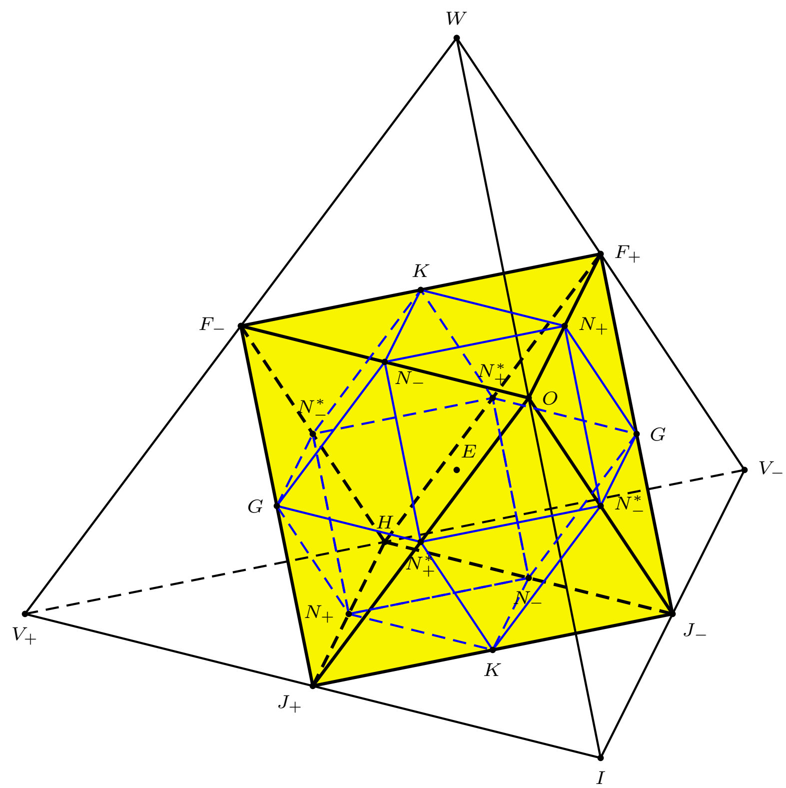

Figure 4

So we have 6 collinearites indexed by edges of the tetrahedron, 3 more by its diagonals, 12 by the inscribed octahedron, and 4 more by the further inscribed cubo-octahedron, for a total of 25.

5. Concluding remarks

In a sense, the location of each center entering into this graphical scheme places it in equal importance to the other centers in similar locations. So the four centers I, U, V, and W, which arose as perspectors with the excentral triangle are on one level. On the next level we may place the six centers O, H, J_+, J_-, F_+, and F_- which index the edges of the tetrahedron and the vertices of the inscribed octahedron. It is interesting that these six centers are the first to appear in the construction given by the author [1], and that the subsequent centers indexed by the midpoints of the edges of the octahedron arise as intersections of lines they determine. The Euler infinity point, E, is the only point at the third level of construction. Centers I, V_+, V_-, W, O, H, F_+ , J_+, F_-, J_-, and E all lie on Neuberg's cubic curve. The Euler line appears as the collinearity $\mathcal{L}(O, E, H)$, with no indication that G lies on the line. The Brocard axis appears four times as $\mathcal{L}(J_+, K, J_-)$, $\mathcal{L}(K, N_-^*, N_+^*)$, $\mathcal{L}(O, N_+^*, J_+)$, and $\mathcal{L}(O, N_-^*, J-)$, but the better-known collinearity $\mathcal{L}(O, J_+, J_-)$ does not.

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[3] C. Kimberling, *Encyclopedia of Triangle Centers*, August 16, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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