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The Apollonius Circle and Related Triangle Centers

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Abstract. We give a simple construction of the Apollonius circle without directly invoking the excircles. This follows from a computation of the coordinates of the centers of similitude of the Apollonius circle with some basic circles associated with a triangle. We also find a circle orthogonal to the five circles, circumcircle, nine-point circle, excentral circle, radical circle of the excircles, and the Apollonius circle.

1. The Apollonius circle of a triangle

The Apollonius circle of a triangle is the circle tangent internally to each of the three excircles. Yiu [5] has given a construction of the Apollonius circle as the inversive image of the nine-point circle in the radical circle of the excircles, and the coordinates of its center Q. It is known that this radical circle has center the Spieker center S and radius $\rho = \frac{1}{2}\sqrt{r^2 + s^2}$. See, for example, [6, Theorem 4]. Ehrmann [1] found that this center can be constructed as the intersection of the Brocard axis and the line joining S to the nine-point center N. See Figure 1. A proof of this fact was given in [2], where Grinberg and Yiu showed that the Apollonius circle is a

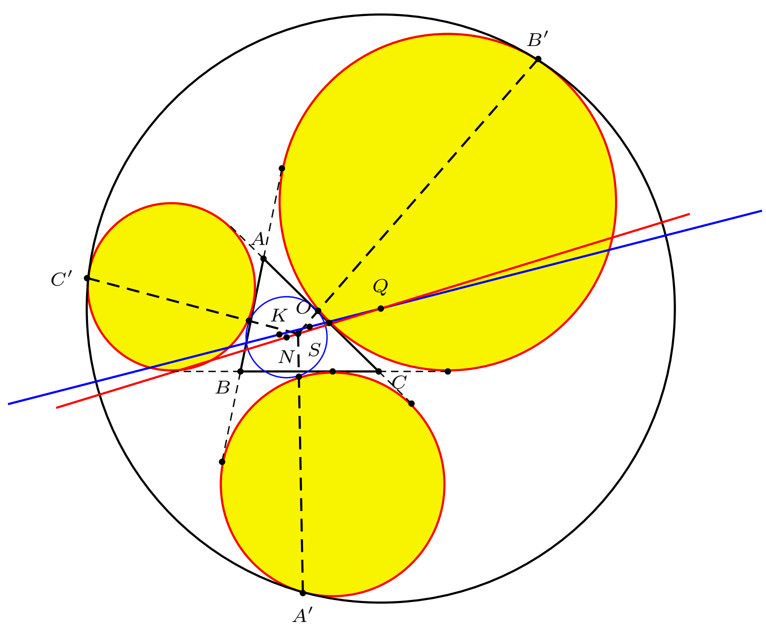

Figure 1

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Tucker circle. In this note we first verify these results by expressing the coordinates of Q in terms of R, r, and s, (the circumradius, inradius, and semiperimeter) of the triangle. By computing some homothetic centers of circles associated with the Apollonius circle, we find a simple construction of the Apollonius circle without directly invoking the excircles. See Figure 4.

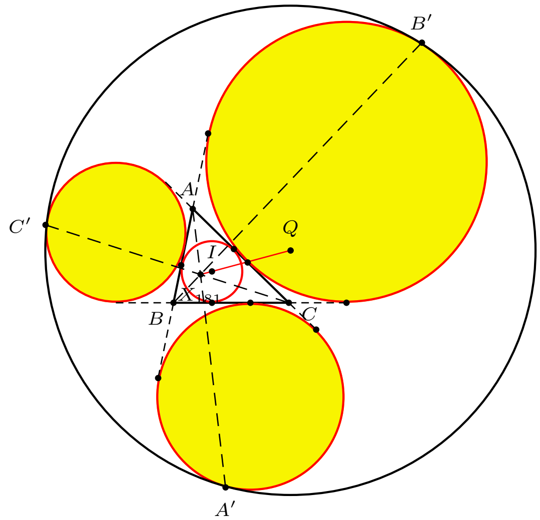

Figure 2

For triangle centers we shall adopt the notation of Kimberling's *Encyclopedia of Triangle Centers* [3], except for the most basic ones:

G	centroid	O	circumcenter
I	incenter	H	orthocenter
N	nine-point center	K	symmedian point
S	Spieker center	I'	reflection of I in O

We shall work with barycentric coordinates, absolute and homogeneous. It is known that if the Apollonius circle touches the three excircles respectively at A', B', C', then the lines AA', BB', CC' concur in the point [1]

$$X_{181} = \left(\frac{a^2(b+c)^2}{s-a} : \frac{b^2(c+a)^2}{s-b} : \frac{c^2(a+b)^2}{s-c} \right).$$

We shall make use of the following simple lemma.

Lemma 1. *Under inversion with respect to a circle, center P, radius ρ, the image of the circle center P', radius ρ', is the circle, radius $\left| \frac{\rho^2}{d^2-\rho'^2} \cdot \rho' \right|$ and center Q which divides the segment PP' in the ratio*

$$PQ : QP' = \rho^2 : d^2 - \rho^2 - \rho'^2,$$

[1]The trilinear coordinates of X_{181} were given by Peter Yff in 1992.

where d is the distance between P and P'. Thus,

$$Q = \frac{(d^2 - \rho^2 - \rho'^2)P + \rho^2 \cdot P'}{d^2 - \rho'^2}.$$

Theorem 2. *The Apollonius circle has center*

$$Q = \frac{1}{4Rr}\left((r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right)$$

and radius $\frac{r^2+s^2}{4r}$.

Proof. It is well known that the distance between O and I is given by

$$OI^2 = R^2 - 2Rr.$$

Since S and N divide the segments IG and OG in the ratio $3 : -1$,

$$SN^2 = \frac{R^2 - 2Rr}{4}.$$

Applying Lemma 1 with

$$\begin{aligned} &P = S = \tfrac{1}{2}(3G - I) = \tfrac{1}{2}(2O + H - I), && P' = N = \tfrac{1}{2}(O + H),\\ &\rho^2 = \tfrac{1}{4}(r^2 + s^2), && \rho'^2 = \tfrac{1}{4}R^2,\\ &d^2 = SN^2 = \tfrac{1}{4}(R^2 - 2Rr), && \end{aligned}$$

we have

$$Q = \frac{1}{4Rr}\left((r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right).$$

The radius of the Apollonius circle is $\frac{r^2+s^2}{4r}$. □

The point Q appears in Kimberling's *Encyclopedia of Triangle Centers* [3] as

$$\begin{aligned} X_{970} =&(a^2(a^3(b+c)^2 + a^2(b+c)(b^2+c^2) - a(b^4 + 2b^3c + 2bc^3 + c^4)\\ &- (b+c)(b^4 + c^4)) : \cdots : \cdots). \end{aligned}$$

We verify that it also lies on the Brocard axis.

Proposition 3.

$$\overrightarrow{OQ} = -\frac{s^2 - r^2 - 4Rr}{4Rr} \cdot \overrightarrow{OK}.$$

Proof. The oriented areas of the triangles KHI, OKI, and OHK are as follows.

$$\begin{aligned} \triangle(KHI) =& \frac{(a-b)(b-c)(c-a)f}{16(a^2+b^2+c^2)\cdot\triangle},\\ \triangle(OKI) =& \frac{abc(a-b)(b-c)(c-a)}{8(a^2+b^2+c^2)\cdot\triangle},\\ \triangle(OHK) =& \frac{-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)}{8(a^2+b^2+c^2)\cdot\triangle}, \end{aligned}$$

where $\triangle$ is the area of triangle ABC and

$$\begin{aligned} f =& a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) + 2abc \\ =& 8rs(2R+r). \end{aligned}$$

Since $abc = 4Rrs$ and $(a+b)(b+c)(c+a) = 2s(r^2+2Rr+s^2)$, it follows that, with respect to OHI, the symmedian point K has homogeneous barycentric coordinates

$$\begin{aligned} & f : 2abc : -2(a+b)(b+c)(c+a) \\ =& 8rs(2R+r) : 8Rrs : -4s(r^2+2Rr+s^2) \\ =& 2r(2R+r) : 2Rr : -(r^2+2Rr+s^2). \end{aligned}$$

Therefore,

$$K = \frac{1}{4Rr+r^2-s^2}\left(2r(2R+r)O + 2Rr\cdot H - (r^2+2Rr+s^2)I\right),$$

and

$$\begin{aligned} \overrightarrow{OK} =& \frac{1}{4Rr+r^2-s^2}\left((r^2+s^2)O + 2Rr\cdot H - (r^2+2Rr+s^2)I\right) \\ =& -\frac{4Rr}{s^2-r^2-4Rr}\cdot\overrightarrow{OQ}. \end{aligned}$$

□

2. Centers of similitude

We compute the coordinates of the centers of similitude of the Apollonius circle with several basic circles. Figure 3 below shows the Apollonius circle with the circumcircle, incircle, nine-point circle, excentral circle, and the radical circle (of the excircles). Recall that the excentral circle is the circle through the excenters of the triangle. It has center I' and radius $2R$.

Lemma 4. *Two circles with centers P, P', and radii ρ, ρ' respectively have internal center of similitude $\dfrac{\rho'\cdot P+\rho\cdot P'}{\rho'+\rho}$ and external center of similitude $\dfrac{\rho'\cdot P-\rho\cdot P'}{\rho'-\rho}$.*

Proposition 5. *The homogeneous barycentric coordinates (with respect to triangle ABC) of the centers of similitude of the Apollonius circle with the various circles are as follows.*

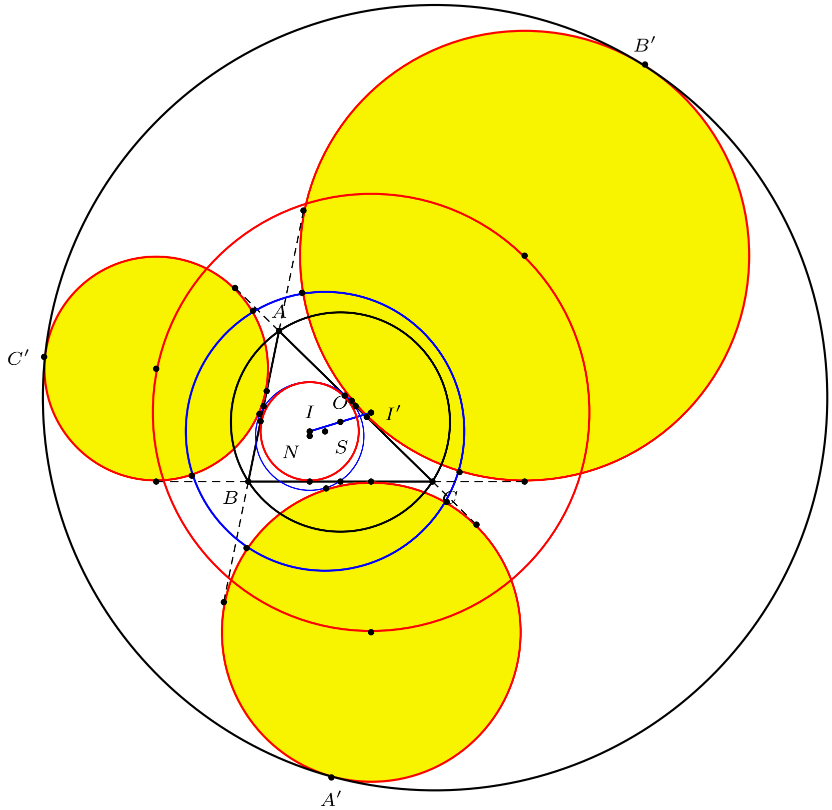

Figure 3

circumcircle	
internal X_{573}	$a^2(a^2(b+c)-abc-(b^3+c^3)) : \cdots : \cdots$
external X_{386}	$a^2(a(b+c)+b^2+bc+c^2) : \cdots : \cdots$
incircle	
internal X_{1682}	$a^2(s-a)(a(b+c)+b^2+c^2)^2 : \cdots : \cdots$
external X_{181}	$\frac{a^2(b+c)^2}{s-a} : \cdots : \cdots$
nine − point circle	
internal S	$b+c : c+a : a+b$
external X_{2051}	$\frac{1}{a^3-a(b^2-bc+c^2)-bc(b+c)} : \cdots : \cdots$
excentral circle	
internal X_{1695}	$a \cdot F : \cdots : \cdots$
external X_{43}	$a(a(b+c)-bc) : \cdots : \cdots$

where

$$\begin{aligned} F = & a^5(b+c)+a^4(4b^2+7bc+4c^2)+2a^3(b+c)(b^2+c^2) \\ & -2a^2(2b^4+3b^3c+3bc^3+2c^4)-a(b+c)(3b^4+2b^2c^2+3c^4)-bc(b^2-c^2)^2. \end{aligned}$$

Proof. The homogenous barycentric coordinates (with respect to triangle OHI) of the centers of similitude of the Apollonius circle with the various circles are as follows.

circumcircle	
internal X_{573} external X_{386}	$2(r^2+2Rr+s^2):2Rr:-(r^2+2Rr+s^2)$ $4Rr:2Rr:-(r^2+2Rr+s^2)$
incircle	
internal X_{1682} external X_{181}	$-r(r^2+4Rr+s^2):-2Rr^2:r^3+Rr^2-(R-r)s^2$ $-r(r^2+4Rr+s^2):-2Rr^2:r^3+3Rr^2+(R+r)s^2$
nine − point circle	
internal S external X_{2051}	$2:1:-1$ $-4Rr:r^2-2Rr+s^2:r^2+2Rr+s^2$
excentral circle	
internal X_{1695} external X_{43}	$4(r^2+2Rr+s^2):4Rr:-(3r^2+4Rr+3s^2)$ $8Rr:4Rr:-(r^2+4Rr+s^2)$

Using the relations

$$r^2=\frac{(s-a)(s-b)(s-c)}{s} \quad \text{and} \quad R=\frac{abc}{4rs},$$

and the following coordinates of O, H, I (with equal coordinate sums),

$$\begin{aligned}
O =&(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)),\\
H =&((c^2+a^2-b^2)(a^2+b^2-c^2), (a^2+b^2-c^2)(b^2+c^2-a^2),\\
&(b^2+c^2-a^2)(c^2+a^2-b^2)),\\
I =&(b+c-a)(c+a-b)(a+b-c)(a,b,c),
\end{aligned}$$

these can be converted into those given in the proposition. □

Remarks. 1. $X_{386}=OK\cap IG$.
2. $X_{573}=OK\cap HI'=OK\cap X_{55}X_{181}$.
3. $X_{43}=IG\cap X_{57}X_{181}$.

From the observation that the Apollonius circle and the nine-point circle have S as internal center of similitude, we have an easy construction of the Apollonius circle without directly invoking the excircles.

Construct the center Q of Apollonius circle as the intersection of OK and NS. Let D be the midpoint of BC. Join ND and construct the parallel to ND through Q (the center of the Apollonius circle) to intersect DS at A', a point on the Apollonius circle, which can now be easily constructed. See Figure 4.

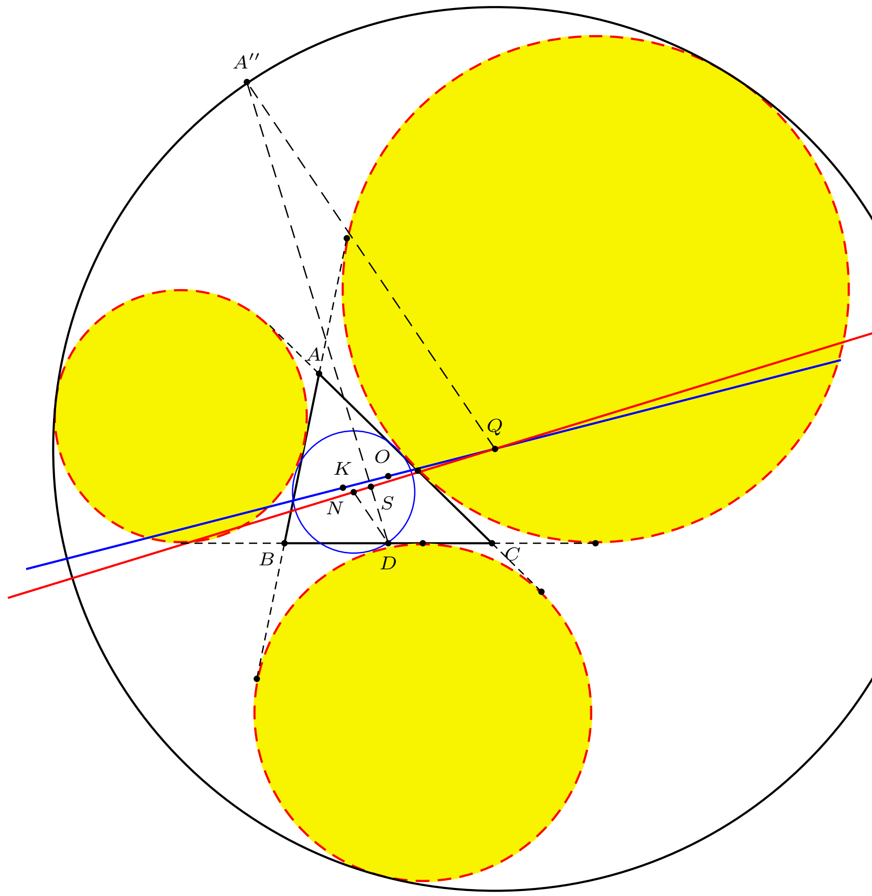

Figure 4

Proposition 6. *The center Q of the Apollonius circle lies on the each of the lines $X_{21}X_{51}$, $X_{40}X_{43}$ and $X_{411}X_{185}$. More precisely,*

$$X_{51}X_{21} : X_{21}Q = 2r : 3R,$$
$$X_{43}X_{40} : X_{43}Q = 8Rr : r^2 + s^2,$$
$$X_{185}X_{411} : X_{411}Q = 2r : R.$$

Remark. The Schiffler point X_{21} is the intersection of the Euler lines of the four triangles ABC, IBC, ICA and IAB. It divides OH in the ratio

$$OX_{21} : X_{21}H = R : 2(R + r).$$

The harmonic conjugate of X_{21} in OH is the triangle center

$$\begin{aligned} X_{411} =&(a(a^6 - a^5(b+c) - a^4(2b^2 + bc + 2c^2) + 2a^3(b+c)(b^2 - bc + c^2) \\ &+ a^2(b^2+c^2)^2 - a(b-c)^2(b+c)(b^2+c^2) + bc(b-c)^2(b+c)^2) \\ &: \cdots : \cdots). \end{aligned}$$

3. A circle orthogonal to 5 given ones

We write the equations of the circles encountered above in the form

$$a^2yz + b^2zx + c^2xy + (x+y+z)L_i = 0,$$

where L_i, $1 \le i \le 5$, are linear forms given below.

i	circle	L_i
1	circumcircle	0
2	nine − point circle	$-\frac{1}{4}((b^2+c^2-a^2)x + (c^2+a^2-b^2)y + (a^2+b^2-c^2)z)$
3	excentral circle	$bcx + cay + abz$
4	radical circle	$(s-b)(s-c)x + (s-c)(s-a)y + (s-a)(s-b)z$
5	Apollonius	$s\left((s+\frac{bc}{a})x + (s+\frac{ca}{b})y + (s+\frac{ab}{c})z\right)$

Remark. The equations of the Apollonius circle was computed in [2]. The equations of the other circles can be found, for example, in [6].

Proposition 7. *The four lines $L_i = 0$, $i = 2,3,4,5$, are concurrent at the point*

$$X_{650} = (a(b-c)(s-a) : b(c-a)(s-b) : c(a-b)(s-c)).$$

It follows that this point is the radical center of the five circles above. From this we obtain a circle orthogonal to the five circles.

Theorem 8. *The circle*

$$a^2yz + b^2zx + c^2xy + (x+y+z)L = 0,$$

where

$$L = \frac{bc(b^2+c^2-a^2)}{2(c-a)(a-b)}x + \frac{ca(c^2+a^2-b^2)}{2(a-b)(b-c)}y + \frac{ab(a^2+b^2-c^2)}{2(b-c)(c-a)}z,$$

is orthogonal to the circumcircle, excentral circle, Apollonius circle, nine-point circle, and the radical circle of the excircles. It has center X_{650} and radius the square root of

$$\frac{abc \cdot G}{4(a-b)^2(b-c)^2(c-a)^2},$$

where

$$\begin{aligned} G =& abc(a^2+b^2+c^2) - a^4(b+c-a) - b^4(c+a-b) - c^4(a+b-c) \\ =& 16r^2s(r^2+5Rr+4R^2-s^2). \end{aligned}$$

This is an interesting result because among these five circles, only three are coaxal, namely, the Apollonius circle, the radical circle, and the nine-point circle.

Remark. X_{650} is also the perspector of the triangle formed by the intersections of the corresponding sides of the orthic and intouch triangles. It is the intersection of the trilinear polars of the Gergonne and Nagel points.

4. More centers of similitudes with the Apollonius circle

We record the coordinates of the centers of similitude of the Apollonius circle with the Spieker radical circle. These are

$$\begin{aligned}&(a^2(-a^3(b+c)^2-a^2(b+c)(b^2+c^2)+a(b^4+2b^3c+2bc^3+c^4)+(b+c)(b^4+c^4))\\&\pm abc(b+c)\sqrt{(b+c-a)(c+a-b)(a+b-c)(a^2(b+c)+b^2(c+a)+c^2(a+b)+abc)}\\&:\cdots:\cdots)\end{aligned}$$

It turns out that the centers of similitude with the Spieker circle (the incircle of the medial triangle) and the Moses circle (the one tangent internally to the nine-point circle at the center of the Kiepert hyperbola) also have rational coordinates in a, b, c:

Spieker circle	
internal	$a(b+c-a)(a^2(b+c)^2+a(b+c)(b^2+c^2)+2b^2c^2)$
external	$a(a^4(b+c)^2+a^3(b+c)(b^2+c^2)-a^2(b^4-4b^2c^2+c^4))$ $-a(b+c)(b^4-2b^3c-2b^2c^2-2bc^3+c^4)+2b^2c^2(b+c)^2)$
Moses circle	
internal	$a^2(b+c)^2(a^3-a(2b^2-bc+2c^2)-(b^3+c^3))$
external	$a^2(a^3(b+c)^2+2a^2(b+c)(b^2+c^2)-abc(b-c)^2$ $-(b-c)^2(b+c)(b^2+bc+c^2))$

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