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Circumrhombi

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Abstract. We consider rhombi circumscribing a given triangle ABC in the sense that one vertex of the rhombus coincides with a vertex of ABC, while the sidelines of the rhombus opposite to this vertex pass through the two remaining vertices of ABC respectively. We construct some new triangle centers associated with these rhombi.

1. Introduction

In this paper we further study the rhombi circumscribing a given reference triangle ABC that the author defined in [4]. These rhombi circumscribe ABC in the sense that each of them shares one vertex with ABC, with its two opposite sides passing through the two remaining vertices of ABC. These rhombi will depend on a fixed angle ϕ and its complement $\overline{\phi} = \frac{\pi}{2} - \phi$. More precisely, for a given ϕ, the rhombus $\mathcal{R}_A(\phi) = AA_cA_aA_b$ will be such that $\angle A_bAA_c = 2\phi$, $B \in A_cA_a$ and $C \in A_bA_a$. Similarly there are rhombi $BB_aB_bB_c$ and $CC_bC_cC_a$.

In [4] it was shown that the vertices of the rhombi opposite to ABC form a triangle $A_aB_bC_c$ perspective to ABC, and that their perspector lies on the Kiepert hyperbola. We give another proof of this result (Theorem 3).

We denote by $\mathcal{K}(\phi) = A^\phi B^\phi C^\phi$ the Kiepert triangle formed by isosceles triangles built on the sides of ABC with base angles ϕ. When the isosceles triangles are constructed outwardly, $\phi > 0$. Otherwise, $\phi < 0$. These vertices have homogeneous barycentric coordinates [1]

$$\begin{aligned}
A^\phi =&(-(S_B + S_C) : S_C + S_\phi : S_B + S_\phi),\\
B^\phi =&(S_C + S_\phi : -(S_C + S_A) : S_A + S_\phi),\\
C^\phi =&(S_B + S_\phi : S_A + S_\phi : -(S_A + S_B)).
\end{aligned}$$

From these it is clear that $\mathcal{K}(\phi)$ is perspective with ABC at the point

$$K(\phi) = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right).$$

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[1]For the notations, see [5].

2. Circumrhombi to a triangle

Theorem 1. *Consider* $\triangle ABC$ *and* $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$. *There are unique rhombi* $\mathcal{R}_A(\phi) = AA_cA_aA_b$, $\mathcal{R}_B(\phi) = BB_aB_bB_c$ *and* $\mathcal{R}_C(\phi) = CC_bC_cC_a$ *with*

$$\angle A_bAA_c = \angle B_cBB_a = \angle C_aCC_b = 2\phi,$$

and $B \in A_cA_a$ *and* $C \in A_bA_a$. *Similarly there are rhombi* $C \in B_aB_b$, $A \in B_cB_b$, $A \in C_bC_c$, $B \in C_aC_c$.

Proof. It is enough to show the construction of $\mathcal{R}_A = \mathcal{R}_A(\phi)$.

Let B_r be the image of B after a rotation through $-2\overline{\phi}$ about A, and C_r the image of C after a rotation through $2\overline{\phi}$ about A. Then let $A_a = B_rC \cap C_rB$. Points $A_c \in C_rA_a$ and $A_b \in B_rA_a$ can be constructed in such a way that $AA_cA_aA_b$ is a parallelogram. Observe that $\triangle AC_rB \equiv \triangle ACB_r$, so that the perpendicular distances from A to lines B_rA_a and C_rA_a are equal. And $AA_cA_aA_b$ must be a rhombus. See Figure 1.

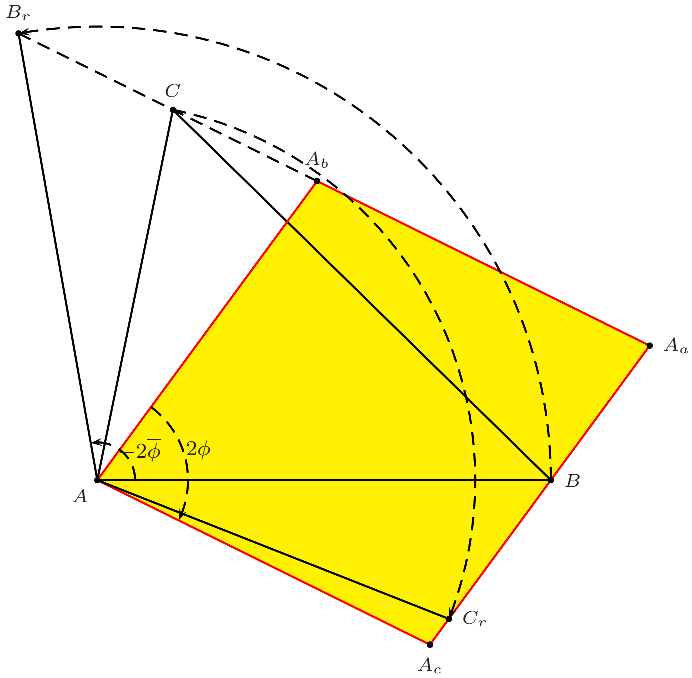

Figure 1

Note that line $B_rC = A_aA_b$ is the image of line $C_rB = A_aA_c$ after rotation through $2\overline{\phi}$ about A, so that the directed angle $\angle A_cA_aA_b = 2\phi$. It follows that $AA_cA_aA_b$ is the rhombus desired in the theorem.

It is easy to see that this is the unique rhombus fulfilling these requirements. When we rotate the complete figure of $\triangle ABC$ and rhombus $AA_cA_aA_b$ through $-2\overline{\phi}$ about A, and let B_r be the image of B again, we see immediately that $B_r \in A_aC$. In the same way we see that the image of C after rotation through $2\overline{\phi}$ about A must be on the line A_aB. $\square$

Consider $\mathcal{R}_A$ and $\mathcal{R}_B$. We note that $\angle AA_aB \equiv \phi \bmod \pi$ and also $\angle AB_bB \equiv \phi \bmod \pi$. This means that ABA_aB_b is cyclic. The center P of its circle should be the apex of the isosceles triangle built on AB such that $\angle APB = 2\phi$, [2] so that $P = C^{\overline{\phi}}$. This shows that $C^{\overline{\phi}}$ lies on the perpendicular bisectors of AA_a and BB_b, hence $A_bA_c \cap B_aB_c = C^{\overline{\phi}}$. See Figure 2.

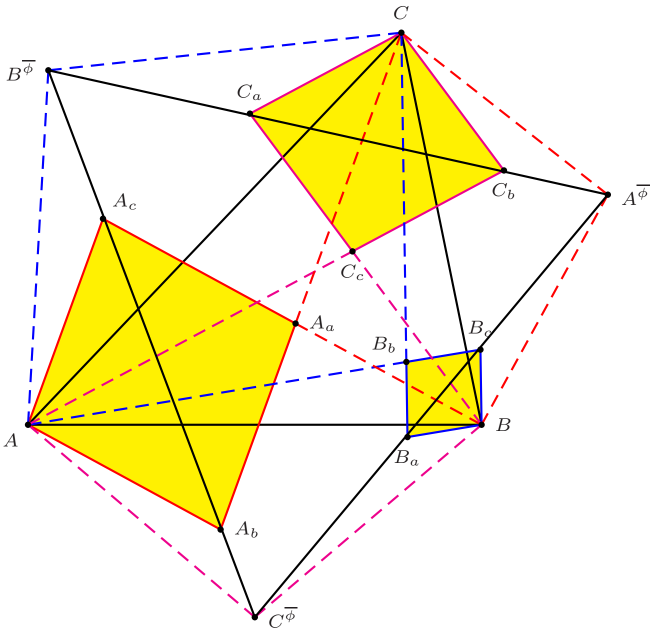

Figure 2

Theorem 2. *The diagonals A_bA_c, B_aB_c and C_aC_b of the circumrhombi $\mathcal{R}_A(\phi)$, $\mathcal{R}_B(\phi)$, $\mathcal{R}_C(\phi)$ bound the Kiepert triangle $\mathcal{K}(\overline{\phi})$.*

3. Radical center of a triad of circles

It is now interesting to further study the circles $A^{\overline{\phi}}(B)$, $B^{\overline{\phi}}(C)$ and $C^{\overline{\phi}}(A)$ with centers at the apices of $\mathcal{K}(\overline{\phi})$, passing through the vertices of ABC. Since the circle $A^{\overline{\phi}}(B)$ passes through B and C, it is represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - kx(x+y+z) = 0.$$

Since it also passes through $A^{-\phi/2} = (-(S_B + S_C) : S_C - S_{\phi/2} : S_B - S_{\phi/2})$, we find

$$k = \frac{S_\phi^2 + 2S_AS_{\phi/2} - S^2}{2S_{\phi/2}} = S_A + S_\phi.$$

[2]Hence, when ϕ is negative, the apex is on the same side of AB as the vertex C.

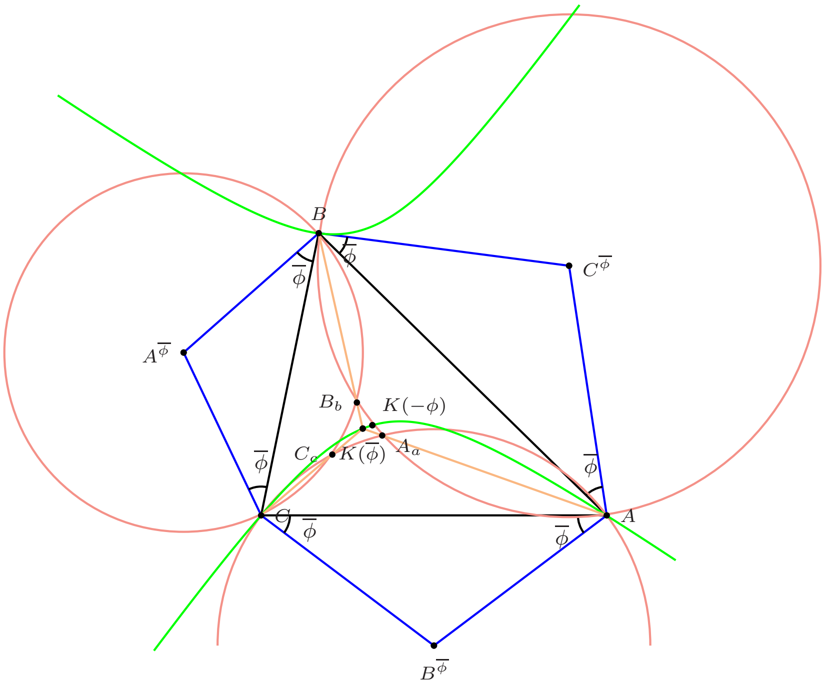

Figure 3

The equations of the three circles are thus

$$\begin{aligned} a^2yz + b^2zx + c^2xy - (S_A + S_\phi)x(x+y+z) &= 0, \\ a^2yz + b^2zx + c^2xy - (S_B + S_\phi)y(x+y+z) &= 0, \\ a^2yz + b^2zx + c^2xy - (S_C + S_\phi)z(x+y+z) &= 0. \end{aligned}$$

From this, it is clear that the radical center of the three circles is the point

$$K(\phi) = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right).$$

The intersections of the circles apart from A, B and C are the points

$$\begin{aligned} A_a &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right), \\ B_b &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi}\right), \\ C_c &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}}\right). \end{aligned} \tag{1}$$

Theorem 3. *The triangle $A_aB_bC_c$ is perspective to ABC and the perspector is $K(\phi)$.*

Remark. For $\phi = \pm\frac{\pi}{3}$, triangle $A_aB_bC_c$ degenerates into the Fermat point $K\left(\pm\frac{\pi}{3}\right)$.

The coordinates of the circumcenter of $A_aB_bC_c$ are too complicated to record here, even in the case of circumsquares. However, we prove the following interesting collinearity.

Theorem 4. *The circumcenters of triangles ABC and $A_aB_bC_c$ are collinear with $K(\phi)$.*

Proof. Since $P = K(\phi)$ is the radical center of $A^{\phi}(B)$, $B^{\phi}(C)$ and $C^{\phi}(A)$ we see that

$$\overline{PA} \cdot \overline{PA_a} = \overline{PB} \cdot \overline{PB_b} = \overline{PC} \cdot \overline{PC_c},$$

which product we will denote by Γ. When $\Gamma > 0$ then the inversion with center P and radius $\sqrt{\Gamma}$ maps A to A_a, B to B_b and C to C_c. Consequently the circumcircles of ABC and $A_aB_bC_c$ are inverses of each other, and the centers of these circles are collinear with the center of inversion.

When $\Gamma < 0$ then the inversion with center P and radius $\sqrt{-\Gamma}$ maps A, B and C to the reflections of A_a, B_b and C_c through P. And the collinearity follows in the same way as above.

When $\Gamma = 0$ the theorem is trivial. $\square$

4. Coordinates of the vertices of the circumrhombi

Along with the coordinates given in (1), we record those of the remaining vertices of the circumrhombi.

$$\begin{aligned}
A_b =&((b^2 + S\csc 2\phi)(S_B + S_\phi) : (S_A - S_{2\phi})(b^2 + S\csc 2\phi) : -(S_A - S_{2\phi})^2),\\
A_c =& \left((c^2 + S\csc 2\phi)(S_C + S_\phi) : -(S_A - S_{2\phi})^2 : (S_A - S_{2\phi})(c^2 + S\csc 2\phi)\right);\\
B_c =&(-(S_B - S_{2\phi})^2 : (c^2 + S\csc 2\phi)(S_C + S_\phi) : (S_B - S_{2\phi})(c^2 + S\csc 2\phi)),\\
B_a =& \left((S_B - S_{2\phi})(a^2 + S\csc 2\phi) : (a^2 + S\csc 2\phi)(S_A + S_\phi) : -(S_B - S_{2\phi})^2\right);\\
C_a =&((S_C - S_{2\phi})(a^2 + S\csc 2\phi) : -(S_C - S_{2\phi})^2 : (a^2 + S\csc 2\phi)(S_A + S_\phi)),\\
C_b =& \left(-(S_C - S_{2\phi})^2 : (S_C - S_{2\phi})(b^2 + S\csc 2\phi) : (b^2 + S\csc 2\phi)(S_B + S_\phi)\right).
\end{aligned} \tag{2}$$

5. The triangle $A'B'C'$

Let $A' = CC_a \cap BB_a$, $B' = CC_b \cap AA_b$ and $C' = AA_c \cap BB_c$. The coordinates of A', using (2), are

$$\begin{aligned}
A' &= \left(a^2 + S\csc 2\phi : -(S_C - S_{2\phi}) : -(S_B - S_{2\phi})\right)\\
&= \left(\frac{a^2 + S\csc 2\phi}{(S_B - S_{2\phi})(S_C - S_{2\phi})} : \frac{-1}{S_B - S_{2\phi}} : \frac{-1}{S_C - S_{2\phi}}\right);
\end{aligned}$$

Similarly for B' and C'. It is clear that $A'B'C'$ is perspective to ABC at $K(-2\phi)$. Note that in absolute barycentric coordinates,

$$\begin{aligned}
&S(\csc 2\phi + 2\cot 2\phi)A' \\
=&\left(a^2 + S\csc 2\phi, -(S_C - S_{2\phi}), -(S_B - S_{2\phi})\right) \\
=&(S_B + S_C, -(S_C + S_{\overline{\phi}}), -(S_B + S_{\overline{\phi}})) + S(\csc 2\phi, \cot 2\phi + \tan\phi, \cot 2\phi + \tan\phi) \\
=&(S_B + S_C, -(S_C + S_{\overline{\phi}}), -(S_B + S_{\overline{\phi}})) + S\csc 2\phi(1,1,1) \\
=&S(-2\tan\phi A^{\overline{\phi}} + 3\csc 2\phi G).
\end{aligned}$$

Now, $\frac{-2\tan\phi}{-2\tan\phi + 3\csc 2\phi} = \frac{4}{1-3\cot^2\phi}$. It follows that

$$A' = \mathsf{h}\left(G, \frac{4}{1-3\cot^2\phi}\right)(A^{\overline{\phi}}).$$

Similarly for B' and C'.

Proposition 5. *Triangles $A'B'C'$ and $\mathcal{K}(\overline{\phi})$ are homothetic at G.*

Corollary 6. *ABC is the Kiepert triangle $\mathcal{K}(-\phi)$ with respect to $A'B'C'$.*

See [5, Proposition 4].

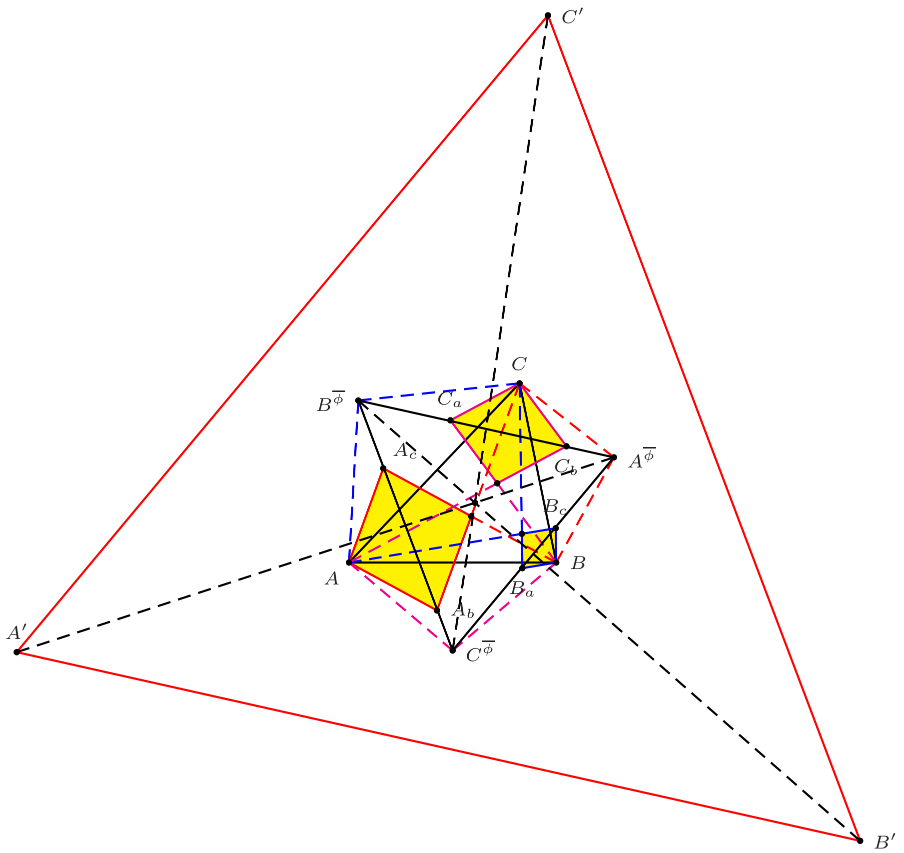

Figure 4

6. The desmic mates

Let XYZ be a triangle perspective with ABC at $P = (u : v : w)$. Its vertices have coordinates

$$X = (x : v : w), \qquad Y = (u : y : w), \qquad Z = (u : v : z),$$

for some x, y, z. The desmic mate of XYZ is the triangle with vertices $X' = BZ \cap CY$, $Y' = CX \cap AZ$, $Z' = AY \cap BX$. These have coordinates

$$X' = (u : y : z), \qquad Y' = (x : v : z), \qquad Z' = (x : y : w).$$

Lemma 7. *The triangle $X'Y'Z'$ is perspective to ABC at $(x : y : z)$ and to XYZ at $(u + x : v + y : w + z)$.*

See, for example, [1, §4].

The desmic mate of $A_aB_bC_a$ has vertices

$$\begin{aligned}
A'_a &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C - S_{2\phi}}\right), \\
B'_b &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}}\right), \qquad (3) \\
C'_c &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi}\right).
\end{aligned}$$

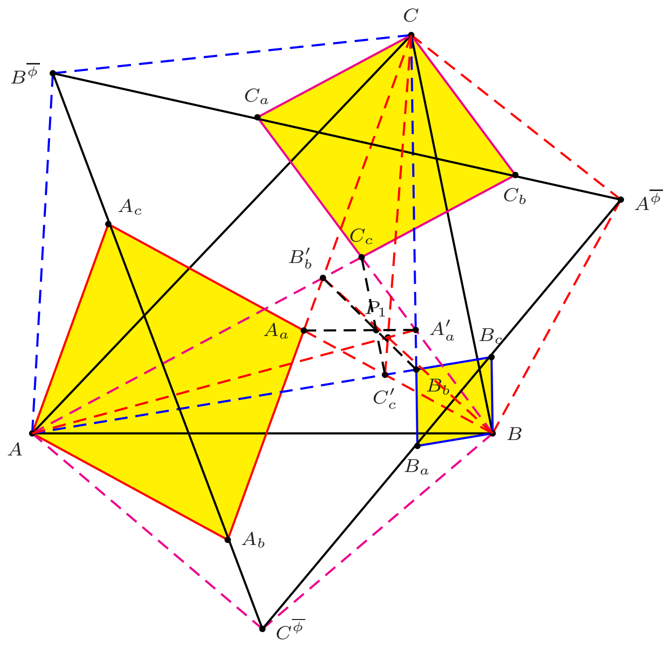

Figure 5

Proposition 8. *Triangle $A'_aB'_bC'_c$ is perspective to ABC at $K(-2\phi)$. It is also perspective to $A_aB_bC_c$ at*

$$P_1(\phi) = \left(\frac{2S_A + S\csc 2\phi}{(S_A+S_\phi)(S_A+S_{2\phi})} : \frac{2S_B + S\csc 2\phi}{(S_B+S_\phi)(S_B+S_{2\phi})} : \frac{2S_C + S\csc 2\phi}{(S_C+S_\phi)(S_C+S_{2\phi})}\right).$$

See Figure 5.

The desmic mate of $A'B'C'$ has vertices

$$\begin{aligned}
A'' =&(-(S_B - S_{2\phi})(S_C - S_{2\phi}) : (S_B - S_{2\phi})(b^2 + S\csc 2\phi)\\
&: (S_C - S_{2\phi})(c^2 + S\csc 2\phi));\\
B'' =&((S_A - S_{2\phi})(a^2 + S\csc 2\phi) : -(S_C - S_{2\phi})(S_A - S_{2\phi})\\
&: (S_C - S_{2\phi})(c^2 + S\csc 2\phi)),\\
C'' =&((S_A - S_{2\phi})(a^2 + S\csc 2\phi) : (S_B - S_{2\phi})(b^2 + S\csc 2\phi)\\
&: -(S_A - S_{2\phi})(S_B - S_{2\phi})).
\end{aligned} \tag{4}$$

Proposition 9. *Triangle $A''B''C''$ is perspective to*

(1) *ABC at*

$$P_2(\phi) = \big((S_A - S_{2\phi})(a^2 + S\csc 2\phi) : \cdots : \cdots\big),$$

(2) *$A'B'C'$ at*

$$P_3(\phi) = ((a^2S_A - S_{BC}) - S\csc 2\phi(S_A - S_\phi\cos 2\phi) : \cdots : \cdots),$$

(3) *the dilated triangle* [3] *at*

$$P_4(\phi) = (S_B + S_C - S_A - S_{2\phi} : \cdots : \cdots).$$

Proof. (1) is clear from the coordinates given in (4). Since

$$\begin{aligned}
&(a^2 + S\csc 2\phi)(S_A - S\cot 2\phi) - (S_B - S\cot 2\phi)(S_C - S\cot 2\phi)\\
=&(a^2S_A - S_{BC}) + S^2\csc 2\phi\cot A - S\cot 2\phi(a^2 + S\csc 2\phi - (S_B+S_C) + S\cot 2\phi)\\
=&(a^2S_A - S_{BC}) + S^2\csc 2\phi\cot A - S^2\cot 2\phi\cot\phi\\
=&(a^2S_A - S_{BC}) - S_AS\csc 2\phi + S_{2\phi}S_\phi\\
=&(a^2S_A - S_{BC}) - S\csc 2\phi(S_A - S_\phi\cos 2\phi),
\end{aligned}$$

it follows from Lemma 7 that $A''B''C''$ is perspective to $A'B'C'$ at

$$\begin{aligned}
&\left(\frac{a^2 + S\csc 2\phi}{(S_B - S_{2\phi})(S_C - S_{2\phi})} - \frac{1}{S_A - S_{2\phi}} : \cdots : \cdots\right)\\
=&((a^2S_A - S_{BC}) - S\csc 2\phi(S_A - S_\phi\cos 2\phi) : \cdots : \cdots).
\end{aligned}$$

[3] This is also called the anticomplementary triangle, it is formed by the lines through the vertices of ABC, parallel to the corresponding opposite sides.

This proves (2). For (3), we rewrite the coordinates for A'' as

$$\begin{aligned} A'' =&(S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1) \\ &+ (S \csc(2\phi) + S_{2\phi} + S_A) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi}) \\ =&(S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1) \\ &+ (S_A + S_\phi) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi}) \end{aligned}$$

From this we see that A'' is on the line connecting the A-vertices of the dilated triangle and the cevian triangle of the isotomic conjugate of $K(-2\phi)$, namely, the point

$$K^\bullet(-2\phi) = (S_A - S_{2\phi} : S_B - S_{2\phi} : S_C - S_{2\phi}).$$

This shows that $A''B''C''$ is perspective to both triangles, and that the perspector is the *cevian quotient* $K^\bullet(-2\phi)/G$, [4] where G denotes the centroid. It is easy to see that this is the superior of $K^\bullet(-2\phi)$. Equivalently, it is $K^\bullet(-2\phi)$ of the dilated triangle, with coordinates

$$(S_B + S_C - S_A - S_{2\phi} : \cdots : \cdots).$$

□

We conclude with a table showing the triangle centers associated with the circumsquares, when $\phi = \pm\frac{\pi}{4}$.

k	$P_k(\frac{\pi}{4})$	$P_k(-\frac{\pi}{4})$
1	$K(\frac{\pi}{4})$	$K(-\frac{\pi}{4})$
2	circumcenter	circumcenter
3	de Longchamps point	de Longchamps point
4	X_{193}	X_{193}

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[4] The cevian quotient X/Y is the perspector of the cevian triangle of X and the precevian triangle of Y. This is the X-Ceva conjugate of Y in the terminology of [2].