

On the Areas of the Intouch and Extouch Triangles

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Abstract. We prove an interesting relation among the areas of the triangles whose vertices are the points of tangency of the sidelines with the incircle and excircles.

1. The intouch and extouch triangles

Consider a triangle ABC with incircle touching the sides BC, CA, AB at A_0, B_0, C_0 respectively. The triangle $A_0B_0C_0$ is called the intouch triangle of ABC . Likewise, the triangle formed by the points of tangency of an excircle with the sidelines is called an extouch triangle. There are three of them, the A -, B -, C -extouch triangles,¹ as indicated in Figure 1. For $i = 0, 1, 2, 3$, let T_i denote the area of triangle $A_iB_iC_i$. In this note we present two proofs of a simple interesting relation among the areas of these triangles.

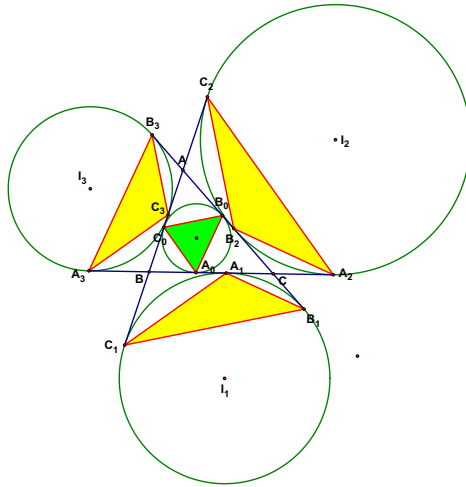


Figure 1

Theorem 1. $\frac{1}{T_0} = \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3}$.

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¹These qualified extouch triangles are not the same as the extouch triangle in [2, §6.9], which means triangle $A_1B_2C_3$ in Figure 1. For a result on this unqualified extouch triangle, see §3.

Proof. Let I be the incenter and r the inradius of triangle ABC . Consider the excircle on the side BC , with center I_1 , tangent to the lines BC , CA , AB at A_1 , B_1 , C_1 respectively. See Figure 2. It is easy to see that triangles $I_1A_1C_1$ and BA_0C_0 are similar isosceles triangles; so are triangles $I_1A_1B_1$ and CA_0B_0 . From these, it easily follows that the angles $B_0A_0C_0$ and $B_1I_1C_1$ are supplementary. It follows that

$$\frac{T_0}{T_1} = \frac{A_0B_0 \cdot A_0C_0}{A_1B_1 \cdot A_1C_1} = \frac{IC}{I_1C} \cdot \frac{IB}{I_1B} = \frac{IB \cdot IC}{I_1B \cdot I_1C}.$$

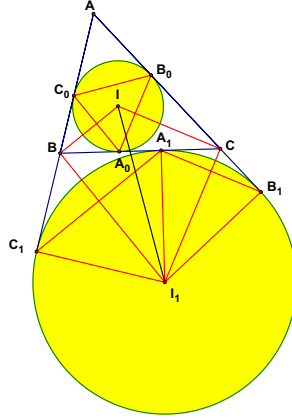


Figure 2

Now, in the cyclic quadrilateral IBI_1C with diameter II_1 ,

$$IB \cdot IC = IB \cdot II_1 \sin \angle I_1IC = II_1 \cdot IA_0 = r \cdot II_1.$$

Similarly, $I_1B \cdot I_1C = II_1 \cdot r_1$, where r_1 is the radius of the A -excircle. It follows that

$$\frac{T_0}{T_1} = \frac{r}{r_1}. \quad (1)$$

Likewise, $\frac{T_0}{T_2} = \frac{r}{r_2}$ and $\frac{T_0}{T_3} = \frac{r}{r_3}$, where r_2 and r_3 are respectively the radii of the B - and C -excircles. From these,

$$\frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} = \left(\frac{r}{r_1} + \frac{r}{r_2} + \frac{r}{r_3} \right) \frac{1}{T_0} = \frac{1}{T_0},$$

since $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$. \square

Corollary 2. Let $ABCD$ be a quadrilateral with an incircle $I(r)$ tangent to the sides at W , X , Y , Z . If the excircles $I_W(r_W)$, $I_X(r_X)$, $I_Y(r_Y)$, $I_Z(r_Z)$ have areas T_W , T_X , T_Y , T_Z respectively, then

$$\frac{T_W}{r_W} + \frac{T_Y}{r_Y} = \frac{T_X}{r_X} + \frac{T_Z}{r_Z} = \frac{T}{r},$$

where T is the area of the intouch quadrilateral $WXYZ$. See Figure 3.

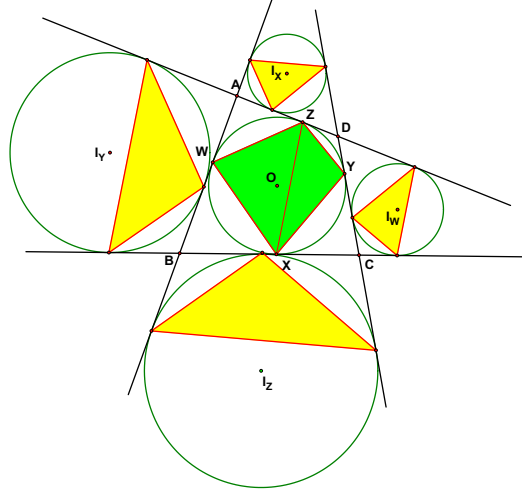


Figure 3

Proof. By (1) above, we have $\frac{T_W}{r_W} = \frac{\text{Area } XYZ}{r}$ and $\frac{T_Y}{r_Y} = \frac{\text{Area } ZWX}{r}$ so that

$$\frac{T_W}{r_W} + \frac{T_Y}{r_Y} = \frac{\text{Area } XYZ + \text{Area } ZWX}{r} = \frac{T}{r}.$$

Similarly, $\frac{T_X}{r_X} + \frac{T_Z}{r_Z} = \frac{T}{r}$. □

2. An alternative proof using barycentric coordinates

The area of a triangle can be calculated easily from its barycentric coordinates. Denote by Δ the area of the reference triangle ABC . The area of a triangle with vertices $A' = (x_1 : y_1 : z_1)$, $B' = (x_2 : y_2 : z_2)$, $C' = (x_3 : y_3 : z_3)$ is given by

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta}{(x_1 + y_1 + z_1)(x_2 + y_2 + z_2)(x_3 + y_3 + z_3)}. \quad (2)$$

Note that this area is signed. It is positive or negative according as triangle $AB'C'$ has the same or opposite orientation as the reference triangle. See, for example, [3]. In particular, the area of the cevian triangle of a point with coordinates $(x : y : z)$ is

$$\frac{\begin{vmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{vmatrix} \Delta}{(y+z)(z+x)(z+y)} = \frac{2xyz\Delta}{(y+z)(z+x)(z+y)}. \quad (3)$$

Let s denote the semiperimeter of triangle ABC , i.e., $s = \frac{1}{2}(a + b + c)$.

The barycentric coordinates of the vertices of the intouch triangle are

$$A_0 = (0 : s - c : s - b), \quad B_0 = (s - c : 0 : s - a), \quad C_0 = (s - b : s - a : 0). \quad (4)$$

The area of the intouch triangle is

$$\begin{aligned} T_0 &= \frac{1}{abc} \begin{vmatrix} 0 & s-c & s-b \\ s-c & 0 & s-a \\ s-b & s-a & 0 \end{vmatrix} \Delta \\ &= \frac{2(s-a)(s-b)(s-c)}{abc} \Delta. \end{aligned}$$

For the A -extouch triangle $A_1B_1C_1$,

$$A_1 = (0 : s-b : s-c), \quad B_1 = (-(s-b) : 0 : s), \quad C_1 = (-(s-c) : s : 0), \quad (5)$$

the area is

$$\frac{1}{abc} \begin{vmatrix} 0 & s-b & s-c \\ -(s-b) & 0 & s \\ -(s-c) & s & 0 \end{vmatrix} \Delta = \frac{-2s(s-b)(s-c)}{abc} \Delta.$$

Similarly, the areas of the B - and C -extouch triangles are $\frac{-2s(s-c)(s-a)}{abc} \Delta$ and $\frac{-2s(s-a)(s-b)}{abc} \Delta$ respectively. Note that these are all negative. Disregarding signs, we have

$$\begin{aligned} \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} &= \frac{abc}{2s(s-a)(s-b)(s-c)} ((s-a) + (s-b) + (s-c)) \cdot \frac{1}{\Delta} \\ &= \frac{abc}{2(s-a)(s-b)(s-c)} \cdot \frac{1}{\Delta} \\ &= \frac{1}{T_0}. \end{aligned}$$

3. A generalization

Using the area formula (3) it is easy to see that the (unqualified) extouch triangle $A_1B_2C_3$ has the same area T_0 as the intouch triangle. This is noted, for example, in [1]. The use of coordinates in §2 also leads to a more general result. Replace the incircle by the inscribed conic with center $P = (p : q : r)$, and the excircles by those with centers

$$P_1 = (-p : q : r), \quad P_2 = (p : -q : r), \quad P_3 = (p : q : -r),$$

respectively. These are the vertices of the anticevian triangle of P , and the four inscribed conics are homothetic. See Figure 4. The coordinates of their points of tangency with the sidelines can be obtained from (4) and (5) by replacing a, b, c by p, q, r respectively. It follows that the areas of intouch and extouch triangles for these conics bear the same relation given in Theorem 1.

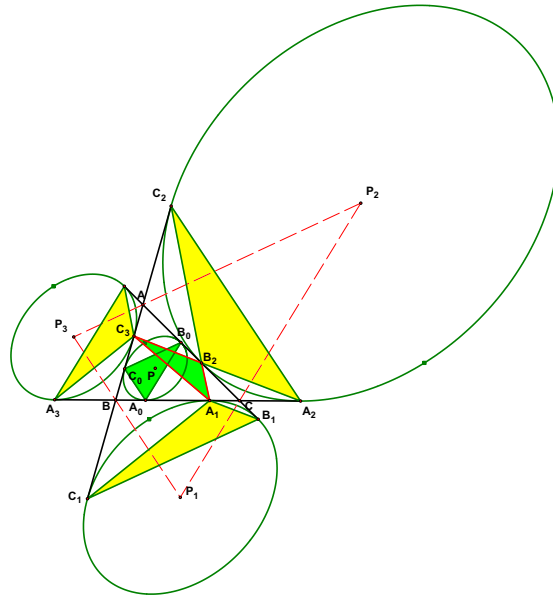


Figure 4

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