Forum Geometricorum
Volume 4 (2004) 69–71.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of the Congruent Isoscelizers Point

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Abstract. We give a very simple construction of the congruent isoscelizers point as an application of the cevian nest theorem.

1. Construction of the congruent isoscelizers point

Given a triangle, an isoscelizer is a segment intercepted in the interior of the triangle by a line perpendicular to an angle bisector. There is a unique point through which the three isoscelizers have equal lengths. This is the congruent isoscelizers points X_{173} of [4]. In this note we present a very simple construction of this triangle center.

Theorem 1. *Let $A'B'C'$ be the intouch triangle of ABC, and $A''B''C''$ the intouch triangle of $A'B'C'$. The triangles $A''B''C''$ and ABC are perspective at the congruent isoscelizers point of ABC.*

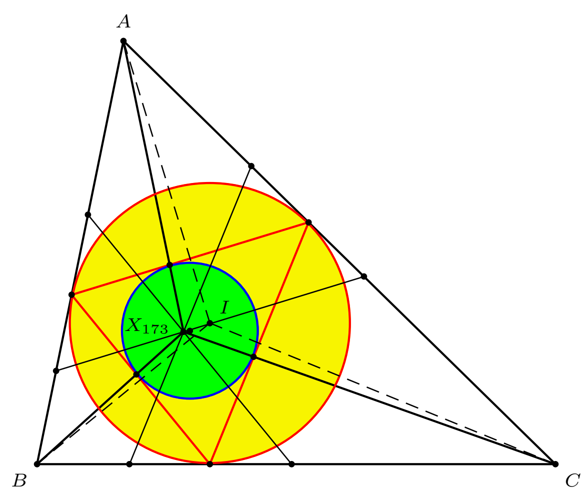

Figure 1

The proof is a simple application of the following cevian nest theorem.[1]

Theorem 2. *Let $A'B'C'$ be the cevian triangle of P in triangle ABC with homogeneous barycentric coordinates $(u : v : w)$ with respect to ABC, and $A''B''C''$*

Publication Date: May 12, 2004. Communicating Editor: Paul Yiu.

[1]Theorem 2 appears in [1, p.165, Supplementary Exercise 7] as follows: The triangle $(Q) = DEF$ is inscribed in the triangle $(P) = ABC$, and the triangle $(K) = KLM$ is inscribed in (Q). Show that if any two of these triangles are perspective to the third, they are perspective to each other.

the cevian triangle of Q in triangle $A'B'C'$, with homogeneous barycentric coordinates $(x : y : z)$ with respect to triangle $A'B'C'$. Triangle $A''B''C''$ is the cevian triangle of

$$Q(P) = \left(\frac{u(v+w)}{x} : \frac{v(w+u)}{y} : \frac{w(u+v)}{z}\right) \tag{1}$$

with respect to triangle ABC.

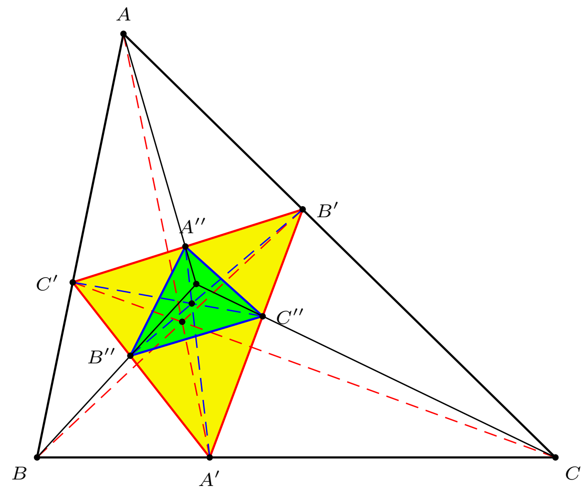

Figure 2

The concurrency of the lines AA'', BB'', CC'' follows from the fact every cevian triangle and every anticevian triangle with respect to $A'B'C'$ are perspective. See, for example, [3, §2.12]. The cevian and anticevian triangles in question are $A''B''C''$ and ABC respectively.

Proof. We compute the absolute barycentric coordinates explicitly.

$$\begin{aligned} A'' = & \frac{yB' + zC'}{y+z} = \frac{y \cdot \frac{wC+uA}{w+u} + z \cdot \frac{uA+vB}{u+v}}{y+z} \\ = & \frac{(y(u+v) + z(w+u))uA + zv(w+u)B + yw(u+v)C}{(y+z)(w+u)(u+v)}. \end{aligned}$$

It is clear that the line AA'' intersects BCat the point with homogeneous barycentric coordinates

$$(0 : zv(w+u) : yw(u+v)) = \left(0 : \frac{v(w+u)}{y} : \frac{w(u+v)}{z}\right).$$

Similarly, the intersections of BB'' with CA, CC'' with AB are the points

$$\left(\frac{u(v+w)}{x} : 0 : \frac{w(u+v)}{z}\right) \quad \text{and} \quad \left(\frac{u(v+w)}{x} : \frac{v(w+u)}{y} : 0\right)$$

respectively. It is clear that the lines AA'', BB'', CC'' intersect at the point given by (1) above. □

2. Proof of Theorem 1

Let P be the Gergonne point, and $A'B'C'$ the intouch triangle. The sidelengths are in the proportions of

$$B'C' : C'A' : A'B' = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}.$$

If Q is the Gergonne point of $A'B'C'$, then we have

$$Q(P) = \left(a\left(-\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) : \cdots : \cdots\right).$$

This is the point X_{173}, the congruent isoscelizers point.

3. Another example

Let P be the incenter of triangle ABC, with cevian triangle $A'B'C'$, and Q the centroid of $A'B'C'$. Then

$$Q(P) = (a(b+c) : b(c+a) : c(a+b)).$$

This is the triangle center X_{37} of [4].

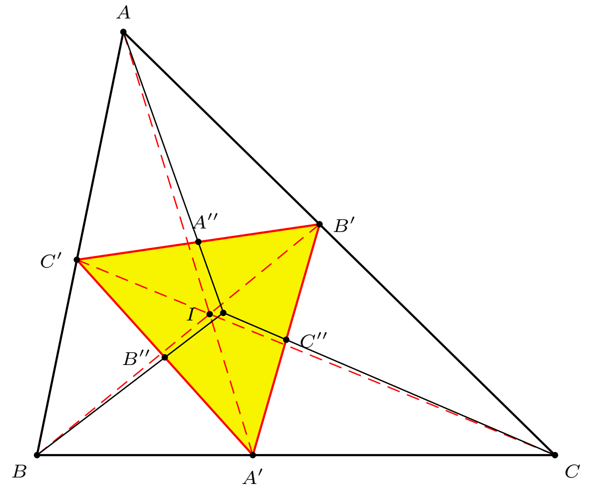

Figure 3

References

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[2] E. Danneels, Hyacinthos message 7892, September 13, 2003.
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[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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