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Triangles with Special Isotomic Conjugate Pairs

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Abstract. We study the condition for the line joining a pair of isotomic conjugates to be parallel to a side of a given triangle. We also characterize triangles in which the line joining a specified pair of isotomic conjugates is parallel to a side.

1. Introduction

Two points in the plane of a given triangle ABC are called isotomic conjugates if the cevians through them divide the opposite sides in ratios that are reciprocals to each other. See [3], also [1]. We study the condition for the line joining a pair of isotomic conjugates to be parallel to a side of a given triangle. We also characterize triangles in which the line joining a specified pair of isotomic conjugates is parallel to a side.

2. Some background material

The standard notation is used throughout: a, b, c for the sides or the lengths of BC, CA, AB respectively of triangle ABC. The median and the altitude through A (and their lengths) are denoted by m_a and h_a respectively. We denote the centroid, the incenter, and the circumcenter by G, I, and O respectively.

2.1. *The orthic triangle.* The triangle formed by the feet of the altitudes is called its orthic triangle. It is the cevian triangle of the orthocenter H. Its sides are easily calculated to be the absolute values of $a\cos A$, $b\cos B$, $c\cos C$.

2.2. *The Gergonne and symmedian points.* The Gergonne point Γ is the concurrence point of the cevians that connect the vertices of triangle ABC to the points of contact of the opposite sides with the incircle.

The symmedian point K is the Gergonne point of the tangential triangle which is bounded by the tangents to the circumcircle at A, B, C.

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2.3. *The Brocard points.* The Crelle-Brocard points Ω_+ and Ω_- are the interior points such that

$$\angle\Omega_+AB = \angle\Omega_+BC = \angle\Omega_+CA = \omega,$$
$$\angle\Omega_-AC = \angle\Omega_-BA = \angle\Omega_-CB = \omega,$$

where ω is the Crelle-Brocard angle.

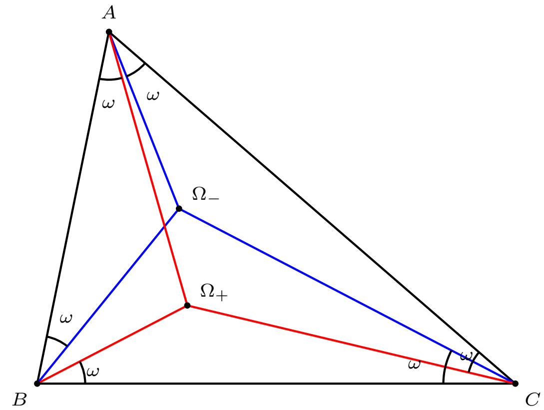

Figure 1

It is known that

$$\cot\omega = \cot A + \cot B + \cot C.$$

See, for example, [3, 5]. According to [4],

$$A + \omega = \frac{\pi}{2} \text{ if and only if } \tan^2 A = \tan B \tan C. \tag{1}$$

2.4. *Self-altitude triangles.* The sides a, b, c of a triangle are in geometric progression if and only if they are proportional to h_a, h_b, h_c in some order. Such a triangle is called a self-altitude triangle in [6]. It has a number of interesting properties. Suppose $a^2 = bc$. Then

(1) Ω_+ and Ω_- are the perpendicular feet of the symmedian point K on the perpendicular bisectors of AC and AB.
(2) The line $\Omega_+\Omega_-$ coincides with the bisector AI.
(3) $B\Omega_+$ and $C\Omega_-$ are tangent to the Brocard circle which has diameter OK.
(4) The median BG and the symmedian CK intersect on AI; so do CG and BK.

See Figure 2.

2.5. *A generalization of a property of equilateral triangles.* An equilateral triangle ABC has this easily provable property: if P is any point on the minor arc BC of the circumcircle of ABC, then $AP = BP+PC$. Surprisingly, however, if triangle ABC is non-isosceles, then there exists a unique point P on the arc BC (not containing the vertex A) such that $AP = BP + PC$ if and only if $a = \frac{mb^2+nc^2}{mb+nc}$.

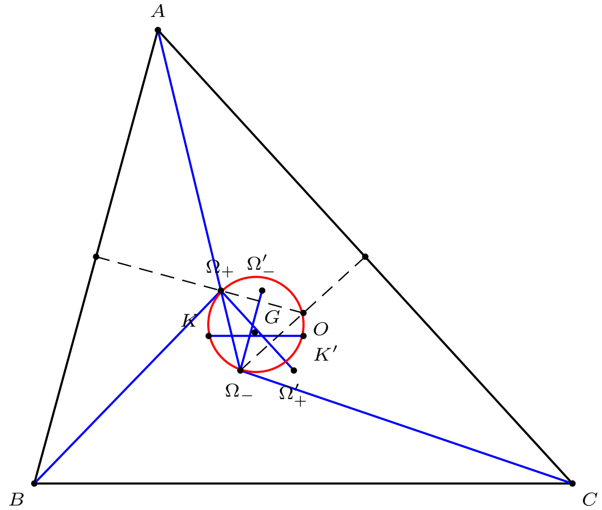

Figure 2

See [8]. Here, $\frac{m}{n}$ is the ratio in which AP divides the side BC. In particular, the extension AP of the median m_a has the preceding property if and only if

$$a = \frac{b^2 + c^2}{b + c}. \tag{2}$$

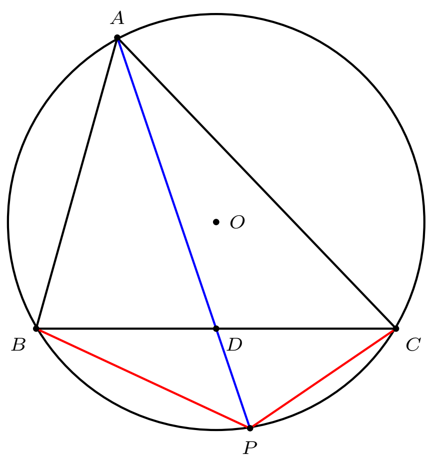

Figure 3.

3. Homogeneous barycentric coordinates

With reference to triangle ABC, every point in the plane is specified by a set of homogeneous barycentric coordinates. See, for example, [9]. If P is a point (not on any of the side lines of triangle ABC) with coordinates $(x : y : z)$, its isotomic conjugate P' has coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. Here are the coordinates of some of classical triangle centers.

Point	Coordinates
centroid G	$(1 : 1 : 1)$
incenter I	$(a : b : c)$
circumcenter O	$(a\cos A : b\cos B : c\cos C)$
orthocenter H	$(\tan A : \tan B : \tan C)$
symmedian point K	$(a^2 : b^2 : c^2)$
Gergonne point Γ	$\left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$
Brocard point Ω_+	$\left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$
Brocard point Ω_-	$\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$

The isotomic conjugate of the Gergonne point is the Nagel point N, which is the concurrence points of the cevians joining the vertices to the point of tangency of its opposite side with the excircle on that side. It has coordinates $(b+c-a : c+a-b : a+b-c)$.

The homogeneous barycentric coordinate of a point can be normalized to give its *absolute* homogeneous barycentric coordinate, provided the sum of the coordinates is nonzero. If $P = (x : y : z)$, we say that in absolute barycentric coordinates,

$$P = \frac{xA + yB + zC}{x + y + z},$$

provided $x + y + z \neq 0$. Points $(x : y : z)$ with $x + y + z = 0$ are called infinite points. The isotomic conjugate of $P = (x : y : z)$ is an infinite point if and only if $xy + yz + zx = 0$. This is the Steiner circum-ellipse which has center at the centroid G of triangle ABC. Another fruitful way is to view an infinite point as the difference $Q - P$ of the absolute barycentric coordinates of two points P and Q. As such, it represents the vector $\overrightarrow{PQ}$.

4. The basic results

The segment joining P to its isotomic conjugate is represented by the infinite point

$$\begin{aligned} PP' =& \frac{yzA + zxB + xyC}{xy + yz + zx} - \frac{xA + yB + zC}{x + y + z} \\ =& \frac{(y+z)(yz - x^2)A + (z+x)(zx - y^2)B + (x+y)(xy - z^2)C}{(x+y+z)(xy+yz+zx)}. \end{aligned} \tag{3}$$

This is parallel to the line BC if it is a multiple of the infinte point of BC, namely, $-B + C$. This is the case if and only if

$$(y + z)(x^2 - yz) = 0. \tag{4}$$

The equation $y + z = 0$ represents the line through A parallel to BC. It is clear that this line is invariant under isotomic conjugation. Every finite point on this line has coordinates $(x : 1 : -1)$ for a nonzero x. Its isotomic conjugate is the point $(\frac{1}{x} : 1 : -1)$ on the same line. On the other hand, the equation $x^2 - yz = 0$ represent an ellipse homothetic to the Steiner circum-ellipse. It passes through $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, $G = (1 : 1 : 1)$, and $(-1 : 1 : 1)$. It is tangent to AB and AC at B and C respectively. It is obtained by translating the Steiner circum-ellipse along the vector $\overrightarrow{AG}$. We summarize this in the following theorem.

Theorem 1. *Let P be a finite point. The line joining P to its isotomic conjugate if parallel to BC if and only if P lies on the line through A parallel to BC or the ellipse through the centroid tangent to AB and AC at B and C respectively. In the latter case, the isotomic conjugate P' is the second intersection of the ellipse with the line through P parallel to BC.*

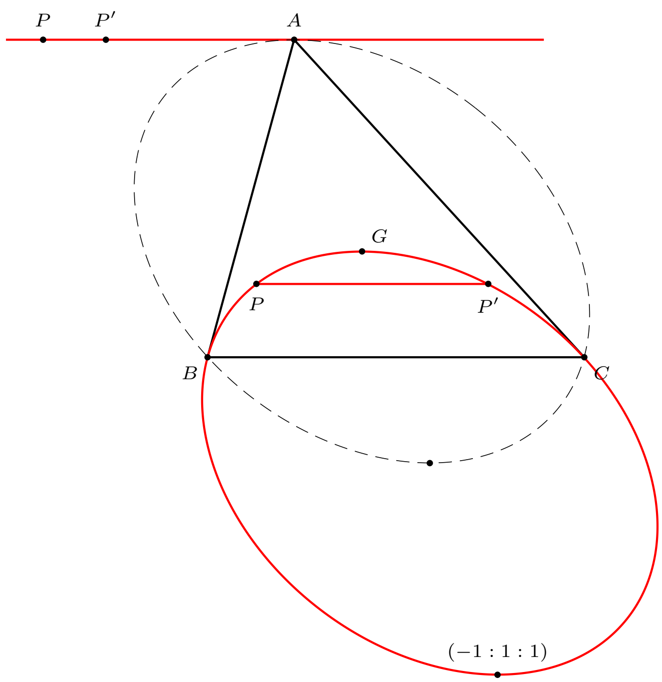

Figure 4

Now we consider the possibility for PP' not only to be parallel to BC, but also equal to one half of its length. This means that the vector PP' is $\pm\frac{1}{2}(C - B)$. If P is a finite point on the parallel to BC through A, we write $P = (x : 1 : -1)$, $x \neq 0$. From (3), we have $PP' = \frac{(1-x^2)(-B+C)}{x} = \frac{1}{2}(-B + C)$ if and only if $x = \frac{-1\pm\sqrt{17}}{4}$. These give the first two pairs of isotomic conjugates listed in Theorem 2 below.

By Theorem 1, P may also lie on the ellipse $x^2 - yz = 0$. It is convenient to use a parametrization

$$x = \mu, \quad y = \mu^2, \quad z = 1. \tag{5}$$

Setting the coefficient of C in (3) to $\frac{1}{2}$, simplifying, we obtain

$$\frac{\mu^2-\mu-3}{2(\mu^2+\mu+1)}=0.$$

The only possibilities are $\mu=\frac{1}{2}\left(1\pm\sqrt{13}\right)$. These give the last two pairs in Theorem 2 below.

Theorem 2. *There are four pairs of isotomic conjugates P, P' for which the segment PP' is parallel to BC and has half of its length.*

i	P_i	P'_i
1	$(\sqrt{17}-1:4:-4)$	$(\sqrt{17}+1:4:-4)$
2	$(\sqrt{17}+1:-4:4)$	$(\sqrt{17}-1:-4:4)$
3	$(\sqrt{13}+1:\ \sqrt{13}+7:\ 2)$	$(\sqrt{13}+1:\ 2:\ \sqrt{13}+7)$
4	$(-(\sqrt{13}-1):\ 7-\sqrt{13}:\ 2)$	$(-(\sqrt{13}-1):\ 2:\ 7-\sqrt{13})$

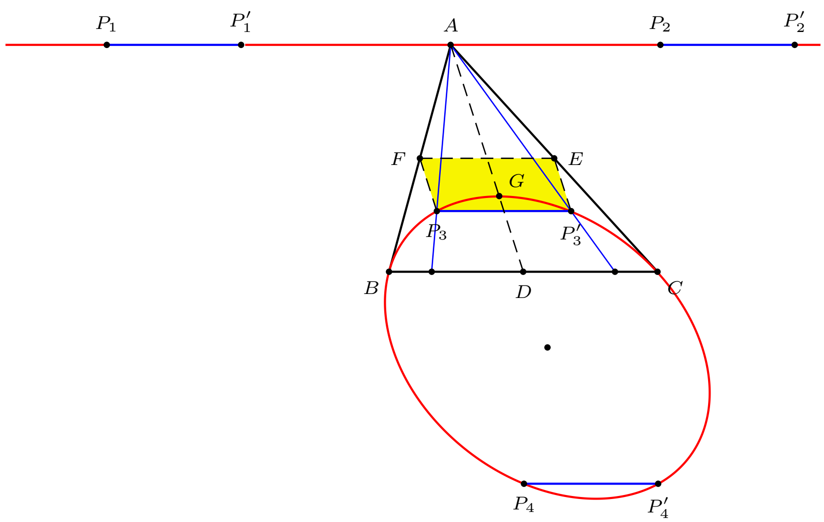

Figure 5

Among these four pairs, only the pair $(P_3,\ P'_3)$ are interior points. The segments FP_3 and EP'_3 are parallel to the median AD, and $P_3P'_3EF$ is a parallelogram with $FP_3=EP'_3=\frac{(5-\sqrt{13})m_a}{6}$.

5. Triangles with specific PP' parallel to BC

We examine the condition under which the line joining a pair of isotomic conjugates is parallel to C. We shall exclude the trivial case of equilateral triangles.

5.1. *The incenter.* Since the incenter has coordinates $(a : b : c)$, if II' is parallel to BC, we must have, according to (5), $a^2 - bc = 0$. Therefore, the triangle is self-altitude. See §2.4. It is, however, not possible to have II' equal to half of the side BC, since the coordinates of P_3 in Theorem 2 do not satisfy the triangle inequality.

5.2. *The symmedian and Brocard points.* Likewise, for the symmedian point K, the line KK' is parallel to BC if and only if $a^4 = b^2c^2$, or $a^2 = bc$. In other words, the triangle is self-altitude again. In fact, the following statements are equivalent.

(1) $a^2 = bc$.
(2) K is on the ellipse $x^2 - yz = 0$; KK' is parallel to BC.
(3) Ω_+ is on the ellipse $z^2 - xy = 0$; $\Omega_+\Omega'_+$ is parallel to CA.
(4) Ω_- is on the ellipse $y^2 - zx = 0$; $\Omega_-\Omega'_-$ is parallel to BA.

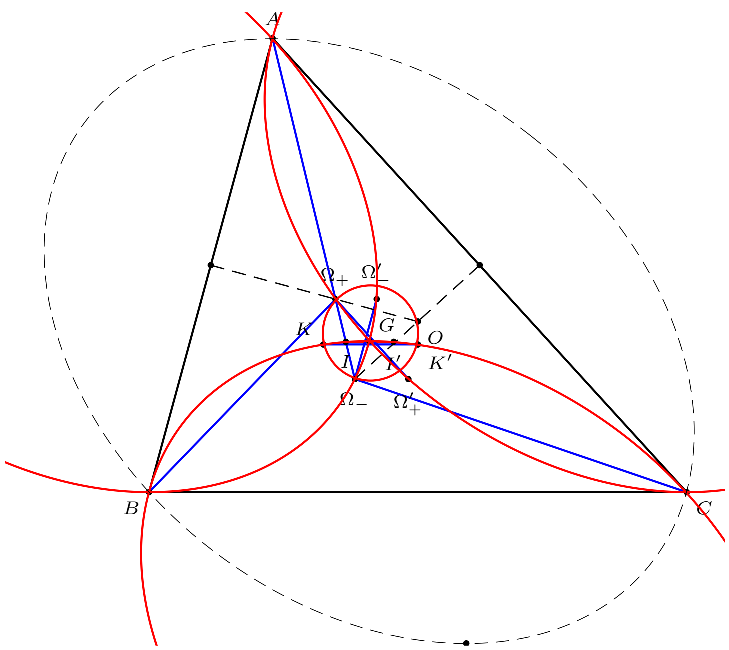

Figure 6

The self-altitude triangle with sides

$$a : b : c = \sqrt{2(1 + \sqrt{13})} : 1 + \sqrt{13} : 2$$

has $KK' = \frac{1}{2}BC$.

5.3. *The circumcenter.* Unlike the incenter, the circumcenter may be outside the triangle. If O lies on the line $y + z = 0$, then $b \cos B + c \cos C = 0$. From this we deduce $\cos(B - C) = 0$, and $|B - C| = \pm\frac{\pi}{2}$. (This also follows from [2] by noting that the nine-point center lies on BC).

The homogeneous barycentric coordinates of the circumcenter are proportional to the sides of the orthic triangle (the pedal triangle of the orthocenter). To construct such a triangle, we take a self-altitude triangle $A'B'C'$ with incenter I_0, and construct the perpendiculars to $I'A'$, $I'B'$, $I'C'$ at A', B', C' respectively. These bound a triangle ABC whose orthocenter is I_0. Its circumcenter O is such that OO' is parallel to BC.

5.4. *The orthocenter.* The orthocenter has barycentric coordinates $(\tan A : \tan B : \tan C)$. If the triangle is acute, the condition $\tan^2 A = \tan B \tan C$ is equivalent to $A + \omega = \frac{\pi}{2}$ according to (1).

5.5. *The Gergonne and Nagel points.* The line joining the Gergonne and Nagel points is parallel to BC if and only if $(b+c-a)^2 = (c+a-b)(a+b-c)$. This is equivalent to (2). Hence, we have a characterization of such a triangle: the extension of the median m_a intersects the minor arc BC at a point P such that $AP = BP + CP$.

Since the Gergonne and Nagel points are interior points, there is a triangle (up to similarity) with ΓN parallel to BC and half in length. From

$$b+c-a : c+a-b : a+b-c = \sqrt{13}+1 : \ 2 : \ \sqrt{13}+7,$$

we obtain

$$a : b : c = \sqrt{13}+9 : 2\sqrt{13}+8 : \sqrt{13}+3 = 3\sqrt{13}-7 : \sqrt{13}+1 : 2.$$

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