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# On the Schiffler center

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**Abstract**. Suppose that $ABC$ is a triangle in the Euclidean plane and $I$ its incenter. Then the Euler lines of $ABC$, $IBC$, $ICA$, and $IAB$ concur at a point $S$, the Schiffler center of $ABC$. In the main theorem of this paper we give a projective generalization of this result and in the final part, we construct Schiffler-like points and a lot of other related centers. Other results in connection with the Schiffler center can be found in the articles [1] and [3].

## 1. Introduction

We recall some formulas and tools of projective geometry, which will be used in §2. Although we focus our attention on the real projective plane, it will be convenient to work in the complex projective plane $\mathcal{P}$.

1.1. Suppose that $(x_1, x_2)$ are projective coordinates on a complex projective line and that two pairs of points are given as follows: $P_1$ and $P_2$ by the quadratic equation

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = 0 \tag{1}$$

and $Q_1$ and $Q_2$ by

$$a'x_1^2 + 2b'x_1x_2 + c'x_2^2 = 0. \tag{2}$$

Then the cross-ratio $(P_1P_2Q_1Q_2)$ equals $-1$ iff

$$ac' - 2bb' + a'c = 0. \tag{3}$$

*Proof.* Put $t = \frac{x_1}{x_2}$ and assume that $t_1, t_2$ ($t_1'$, $t_2'$ respectively) are the solutions of (1) ((2) respectively), divided by $x_2^2$. Then $(t_1\, t_2\, t_1'\, t_2') = -1$ is equivalent to $2(t_1t_2 + t_1't_2') = (t_1 + t_2)(t_1' + t_2')$ or $2(\frac{c}{a} + \frac{c'}{a'}) = (-\frac{2b}{a})(-\frac{2b'}{a'})$, which gives (3). □

1.2.1. Consider a triangle $ABC$ in the complex projective plane $\mathcal{P}$ and assume that $\ell$ is a line in $\mathcal{P}$, not through $A, B$, or $C$. Put $AB \cap \ell = M'_C, BC \cap \ell = M'_A$, and $CA \cap \ell = M'_B$ and determine the points $M_C$, $M_A$, and $M_B$ by $(ABM'_CM_C) = (BCM'_AM_A) = (CAM'_BM_B) = -1$, then $AM_A$, $BM_B$, and $CM_C$ concur at a point $Z$, the so-called trilinear pole of $\ell$ with regard to $ABC$.

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*Proof.* If $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, and $\ell$ is the unit line $x_1 + x_2 + x_3 = 0$, then $M'_C = (1,-1,0)$, $M'_A = (0,1,-1)$, $M'_B = (1,0,-1)$, and $M_C = (1,1,0)$, $M_A = (0,1,1), M_B = (1,0,1)$, and $Z$ is the unit point $(1,1,1)$. □

1.2.2. The trilinear pole $Z_C$ of the unit-line $\ell$ with regard to $ABQ$, where $A = (1,0,0), B = (0,1,0)$, and $Q = (\mathcal{A},\mathcal{B},\mathcal{C})$, has coordinates $(2\mathcal{A}+\mathcal{B}+\mathcal{C},\mathcal{A}+2\mathcal{B}+\mathcal{C},\mathcal{C})$.

*Proof.* The point $Z_C$ is the intersection of the line $QM_C$ and $BM_{QA}$, with $M_C = (1,1,0)$, and $M_{QA}$ the point of $QA$, such that $(Q\ A\ M_{QA}\ M'_{QA}) = -1$, with $M'_{QA} = QA \cap \ell$. We find for $M_{QA}$ the coordinates $(2\mathcal{A}+\mathcal{B}+\mathcal{C},\mathcal{B},\mathcal{C})$, and a straightforward calculation completes the proof. □

1.3. Consider a non-degenerate conic $\mathcal{C}$ in the complex projective plane $\mathcal{P}$, and two points $A, Q$, not on $\mathcal{C}$, whose polar lines with respect to $\mathcal{C}$, intersect $\mathcal{C}$ at $T_1, T_2$, and $I_1, I_2$ respectively. Then $Q$ lies on one of the lines $\ell_1, \ell_2$ through $A$ which are determined by $(AT_1\ AT_2\ \ell_1\ \ell_2) = (AI_1\ AI_2\ \ell_1\ \ell_2) = -1$.

*Proof.* This follows immediately from the fact that the pole of the line AQ with respect to $\mathcal{C}$ is the point $T_1T_2 \cap I_1I_2$. □

1.4. For any triangle $ABC$ of $\mathcal{P}$ and line $\ell$ not through a vertex, the Desargues-Sturm involution theorem ([7, p.341], [8, p.63]) provides a one-to-one correspondence between the involutions on $\ell$ and the points $P$ in $\mathcal{P}$ that lie neither on $\ell$ nor on a side of the triangle. Specifically, the conics of the pencil $\mathcal{B}(A,B,C,P)$ intersect $\ell$ in pairs of points that are interchanged by an involution with fixed points $I$ and $J$. Conversely, $P$ is the fourth intersection point of the conics through $A, B$, and $C$ that are tangent to $\ell$ at $I$ and $J$. The point $P$ can easily be constructed from $A, B, C, I$, and $J$ as the point of intersection of the lines $AA'$, and $BB'$, where $A'$ is the harmonic conjugate of $BC \cap \ell$ with respect to $I$ and $J$, and $B'$ is the harmonic conjugate of $AC \cap \ell$ with respect to $I$ and $J$.

1.5. Denote the pencil of conics through the four points $A_1, A_2, A_3$, and $A_4$ by $\mathcal{B}(A_1,A_2,A_3,A_4)$, and assume that $\ell$ is a line not through $A_i$, $i = 1,\ldots,4$. Put $M'_{12} = A_1A_2 \cap \ell$, and let $M_{12}$ be the harmonic conjugate of $M'_{12}$ with respect to $A_1$ and $A_2$, and define the points $M_{23}$, $M_{34}$, $M_{13}$, $M_{14}$, and $M_{24}$ likewise. Let $X$, $Y$, and $Z$ be the points $A_1A_2 \cap A_3A_4$, $A_2A_3 \cap A_1A_4$, and $A_1A_3 \cap A_2A_4$ respectively. Finally, let $I$ and $J$ be the tangent points with $\ell$ of the two conics of the pencil which are tangent at $\ell$. Then the eleven points $M_{12}$, $M_{13}$, $M_{14}$, $M_{23}$, $M_{24}$, $M_{34}$, $X, Y$, $Z$, $I$, and $J$ belong to a conic ([8, p.109]).

*Proof.* We prove that this conic is the locus $\mathcal{C}$ of the poles of the line $\ell$ with regard to the conics of the pencil $\mathcal{B}(A_1,A_2,A_3,A_4)$. But first, let us prove that this locus is indeed a conic: if we represent the pencil by $F_1 + tF_2 = 0$, where $F_1 = 0$ and $F_2 = 0$ are two conics of the pencil, the equation of the locus is obtained by eliminating $t$ from two linear equations which represent the polar lines of two points of $\ell$, which gives a quadratic equation. Then, call $A'_3$ the point which is the harmonic conjugate of $A_3$ with respect to $M_{12}A_3 \cap \ell$ and $M_{12}$, and consider the conic of the pencil through $A'_3$: the pole of $\ell$ with respect to this conic clearly is $M_{12}$, which means that $M_{12}$, and thus also $M_{ij}$, is a point of the locus. Next, $X$, $Y$ and $Z$ are points of the locus, since they are singular points of the three degenerate conics of the pencil. And finally, $I$ and $J$ belong to the locus, because they are the poles of $\ell$ with regard to the two conics of the pencil which are tangent to $\ell$. □

1.6. Consider again a triangle $ABC$ in $\mathcal{P}$, and a point $P$ not on a side of $ABC$. The *Ceva triangle* of $P$ is the triangle with vertices $AP \cap BC$, $BP \cap CA$, and $CP \cap AB$. Example: with the notation of §1.2.1 the Ceva triangle of $Z$ is $M_A M_B M_C$.

Next, assume that $I$ and $J$ are any two (different) points, not on a side of $ABC$, on a line $\ell$, not through a vertex, and that $P$ is the point which corresponds (according to 1.4) to the involution on $\ell$ with fixed points $I$ and $J$. Let $H'_A H'_B H'_C$ be the Ceva triangle of $P$, let $A'$ ($B'$, and $C'$ respectively) be the harmonic conjugate of $PA \cap \ell$ ($PB \cap \ell$, and $PC \cap \ell$ respectively) with respect to $A$ and $P$ ($B$ and $P$, and $C$ and $P$, respectively), and let $M_A M_B M_C$ be the Ceva triangle of the trilinear pole $Z$ of $\ell$ with regard to $ABC$. Then there is a conic through $I$, $J$, and the triples $H'_A H'_B H'_C$, $A'B'C'$, and $M_A M_B M_C$. This conic is known as the *eleven-point conic* of $ABC$ with regard to $I$ and $J$ ([7, pp.342–343]).

*Proof.* Apply 1.5 to the pencil $\mathcal{B}(A, B, C, P)$. □

## 2. The main theorem

**Theorem.** *Let $ABC$ be a triangle in the complex projective plane $\mathcal{P}$, $\ell$ be a line not through a vertex, and $I$ and $J$ be any two (different) points of $\ell$ not on a side of the triangle. Choose $\mathcal{C}$ to be one of the four conics through $I$ and $J$ that are tangent to the sides of triangle $ABC$, and define $Q$ to be the pole of $\ell$ with respect to $\mathcal{C}$. If $Z$, $Z_A$, $Z_B$, and $Z_C$ are the trilinear poles of $\ell$ with respect to the triangles $ABC$, $QBC$, $QCA$, and $QAB$ respectively, while $P$, $P_A$, $P_B$, and $P_C$ respectively, are the points determined by these triangles and the involution on $\ell$ whose fixed points are $I$ and $J$ (see 1.4), then the lines $PZ$, $P_A Z_A$, $P_B Z_B$, and $P_C Z_C$ concur at a point $S_P$.*

*Proof.* We choose our projective coordinate system in $\mathcal{P}$ as follows : $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, and $\ell$ is the unit line with equation $x_1 + x_2 + x_3 = 0$. The point $P$ has coordinates $(\alpha, \beta, \gamma)$.

Two degenerate conics of the pencil $\mathcal{B}(A, B, C, P)$ are $(CP, AB)$ and $(BP, CA)$, which intersect $\ell$ at the points $(-\alpha, -\beta, \alpha + \beta)$, $(1, -1, 0)$ and $(-\alpha, \alpha + \gamma, -\gamma)$, $(1, 0, -1))$ respectively. Joining these points to $A$, we find the lines $(\alpha + \beta)x_2 + \beta x_3 = 0$, $x_3 = 0$ and $\gamma x_2 + (\alpha + \gamma)x_3 = 0$, $x_2 = 0$, or as quadratic equations $(\alpha + \beta)x_2 x_3 + \beta x_3^2 = 0$ and $\gamma x_2^2 + (\alpha + \gamma)x_2 x_3 = 0$ respectively. Therefore, the lines $AI$ and $AJ$ are given by $kx_2^2 + 2lx_2x_3 + mx_3^2 = 0$ whereby $k$, $l$, and $m$ are solution of (see 1.1):

$$\begin{cases} \beta k - (\alpha + \beta)l = 0 \\ -(\alpha + \gamma)l + \gamma m = 0, \end{cases}$$

and thus $(k,l,m) = (\gamma(\alpha+\beta), \beta\gamma, \beta(\alpha+\gamma))$. Next, the lines through $A$ which form together with $AI$, $AJ$ and with $AB$, $AC$ an harmonic quadruple, are determined by $px_2^2 + 2qx_2x_3 + rx_3^2 = 0$ with $p$, $q$, $r$ solutions of (see again 1.1)

$$\begin{cases} \beta(\alpha+\gamma)p - 2\beta\gamma q + \gamma(\alpha+\beta)r = 0 \\ q = 0, \end{cases}$$

and thus these lines are given by $\gamma(\alpha+\beta)x_2^2 - \beta(\alpha+\gamma)x_3^2 = 0$. In the same way, we find the quadratic equation of the two lines through $B$ ($C$, respectively) which form together with $BI$, $BJ$ and with $BC$, $BA$ (with $CI$, $CJ$ and with $CA$, $CB$ respectively) an harmonic quadruple : $\alpha(\beta+\gamma)x_3^2 - \gamma(\beta+\alpha)x_1^2 = 0$ ($\beta(\gamma+\alpha)x_1^2 - \alpha(\gamma+\beta)x_2^2 = 0$ respectively). The intersection points of these three pairs of lines through $A$, $B$, and $C$ are the poles $Q_1$, $Q_2$, $Q_3$, $Q_4$ of $\ell$ with respect to the four conics through $I$ and $J$ that are tangent to the sides of triangle $ABC$ (see 1.3) and their coordinates are $Q_1(\mathcal{A},\mathcal{B},\mathcal{C})$, $Q_2(-\mathcal{A},\mathcal{B},\mathcal{C})$, $Q_3(\mathcal{A},-\mathcal{B},\mathcal{C})$, and $Q_4(\mathcal{A},\mathcal{B},-\mathcal{C})$, where

$$\mathcal{A} = \sqrt{\alpha(\beta+\gamma)}, \quad \mathcal{B} = \sqrt{\beta(\gamma+\alpha)}, \quad \mathcal{C} = \sqrt{\gamma(\alpha+\beta)}.$$

For now, let us choose for $Q$ the point $Q_1(\mathcal{A},\mathcal{B},\mathcal{C})$.

The coordinates of the points $Z$, $Z_A$, $Z_B$, and $Z_C$ are $(1,1,1)$, $(\mathcal{A}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{A}+\mathcal{B}+2\mathcal{C})$, $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{B}, \mathcal{A}+\mathcal{B}+2\mathcal{C})$, and $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{C})$ (see 1.2.2).

Now, in connection with the point $P_C$, remark that $(AP_C \cap \ell\ (QB \cap \ell)\ I\ J) = -1$. But $(Q_2Q_4 \cap \ell\ (Q_1Q_3 \cap \ell)\ I\ J) = -1$ and $Q_2Q_4 = Q_2B$, $Q_1Q_3 = Q_1B$, so that $AP_C \cap \ell = Q_2B \cap \ell$, and since $Q_2B$ has equation $\mathcal{C}x_1 + \mathcal{A}x_3 = 0$, the point $AP_C \cap \ell$ has coordinates $(\mathcal{A}, \mathcal{C}-\mathcal{A}, -\mathcal{C})$ and the line $AP_C$ has equation $\mathcal{C}x_2 + (\mathcal{C}-\mathcal{A})x_3 = 0$. In the same way, we find the equation of the line $BP_C$: $\mathcal{C}x_1 + (\mathcal{C}-\mathcal{B})x_3 = 0$, and the common point of these two lines is the point $P_C$ with coordinates $(\mathcal{B}-\mathcal{C}, \mathcal{A}-\mathcal{C}, \mathcal{C})$.

Finally, the line $P_CZ_C$ has equation :

$$\mathcal{C}(\mathcal{B}+\mathcal{C})x_1 - \mathcal{C}(A+\mathcal{C})x_2 + (\mathcal{A}^2 - \mathcal{B}^2)x_3 = 0,$$

and cyclic permutation gives us the equations of $P_AZ_A$ and $P_BZ_B$.

Now, $P_AZ_A$, $P_BZ_B$, and $P_CZ_C$ are concurrent if the determinant

$$\begin{vmatrix} \mathcal{B}^2 - \mathcal{C}^2 & \mathcal{A}(\mathcal{C}+\mathcal{A}) & -\mathcal{A}(\mathcal{B}+\mathcal{A}) \\ -\mathcal{B}(\mathcal{C}+\mathcal{B}) & \mathcal{C}^2 - \mathcal{A}^2 & \mathcal{B}(\mathcal{A}+\mathcal{B}) \\ \mathcal{C}(\mathcal{B}+\mathcal{C}) & -\mathcal{C}(\mathcal{A}+\mathcal{C}) & \mathcal{A}^2 - \mathcal{B}^2 \end{vmatrix}$$

is zero, which is obviously the case, since the sum of the rows gives us three times zero. Then, the line $PZ$ has equation $(\beta-\gamma)x_1 + (\gamma-\alpha)x_2 + (\alpha-\beta)x_3 = 0$. But $\mathcal{A}^2 = \alpha(\beta+\gamma)$, $\mathcal{B}^2 = \beta(\gamma+\alpha)$, and $\mathcal{C}^2 = \gamma(\alpha+\beta)$, so that $(\mathcal{B}^2-\mathcal{C}^2)(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2) = 2\alpha\beta\gamma(\beta-\gamma)$, and $PZ$ has also the following equation

$$\begin{aligned} &(\mathcal{B}^2-\mathcal{C}^2)(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2)x_1 + (\mathcal{C}^2-\mathcal{A}^2)(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2)x_2 \\ &+(\mathcal{A}^2-\mathcal{B}^2)(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2)x_3 = 0. \end{aligned}$$

For $PZ$, $P_AZ_A$, and $P_BZ_B$ to be concurrent, the following determinant must vanish :

$$\begin{vmatrix} (\mathcal{B}^2-\mathcal{C}^2)(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2) & (\mathcal{C}^2-\mathcal{A}^2)(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2) & (\mathcal{A}^2-\mathcal{B}^2)(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2) \\ \mathcal{B}^2-\mathcal{C}^2 & \mathcal{A}(\mathcal{C}+\mathcal{A}) & -\mathcal{A}(\mathcal{B}+\mathcal{A}) \\ -\mathcal{B}(\mathcal{C}+\mathcal{B}) & \mathcal{C}^2-\mathcal{A}^2 & \mathcal{B}(\mathcal{A}+\mathcal{B}) \end{vmatrix}$$
$$\begin{aligned}
=&(\mathcal{B}+\mathcal{C})(\mathcal{C}+\mathcal{A})(\mathcal{A}+\mathcal{B})(\mathcal{A}(\mathcal{B}-\mathcal{C})(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2)(-\mathcal{A}+\mathcal{B}+\mathcal{C})\\
&+\mathcal{B}(\mathcal{C}-\mathcal{A})(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2)(\mathcal{A}-\mathcal{B}+\mathcal{C})+\mathcal{C}(\mathcal{A}-\mathcal{B})(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2)(\mathcal{A}+\mathcal{B}-\mathcal{C}))\\
=&0.
\end{aligned}$$

We may conclude that $PZ, P_AZ_A$, $P_BZ_B$, and $P_CZ_C$ are concurrent. This completes the proof. □

*Remarks.* (1) If $Q$ is chosen as the point $Q_2$ ($Q_3$, or $Q_4$, respectively), then $\mathcal{A}$ ($\mathcal{B}$, or $\mathcal{C}$ respectively) must be replaced by $-\mathcal{A}$ ($-\mathcal{B}$, or $-\mathcal{C}$ respectively) in the foregoing proof.

(2) The coordinates of the common point $S_P$ of the lines $PZ, P_AZ_A, P_BZ_B$, and $P_CZ_C$ are $(\mathcal{A}\frac{-\mathcal{A}+\mathcal{B}+\mathcal{C}}{\mathcal{B}+\mathcal{C}}, \mathcal{B}\frac{\mathcal{A}-\mathcal{B}+\mathcal{C}}{\mathcal{C}+\mathcal{A}}, \mathcal{C}\frac{\mathcal{A}+\mathcal{B}-\mathcal{C}}{\mathcal{A}+\mathcal{B}})$.

(3) Of course, when we work in the real (complexified) projective plane $\mathcal{P}$ with a real triangle $ABC$, a real line $\ell$ and a real point $P$, the points $Q$ and $S_P$, are not always real. That depends on the values of $\alpha, \beta$, and $\gamma$ and thus on the position of the point $P$ in the plane. For instance, in example 5.5 of §5, the points $Q$ and $S_P$ will be imaginary.

(4) The conic through $A, B, C$, and through the points $I, J$ on $\ell$ has equation
$$\alpha(\beta+\gamma)x_2x_3+\beta(\gamma+\alpha)x_3x_1+\gamma(\alpha+\beta)x_1x_2=0$$
or
$$\mathcal{A}^2x_2x_3+\mathcal{B}^2x_3x_1+\mathcal{C}^2x_1x_2=0.$$
Indeed, eliminating $x_1$ from this equation and from $x_1+x_2+x_3=0$, gives us $\gamma(\alpha+\beta)x_2^2+2\gamma\beta x_2x_3+\beta(\gamma+\alpha)x_3^2=0$, which determines the lines $AI$ and $AJ$ (see the proof of the theorem).
The pole of the line $\ell$ with respect to this conic is the point $Y(\beta+\gamma, \gamma+\alpha, \alpha+\beta)$, which clearly is a point of the line $PZ$. We denote this conic by $(Y)$.

(5) The locus of the poles of the line $\ell$ with respect to the conics of the pencil $\mathcal{B}(A, B, C, P)$ is the conic with equation
$$\beta\gamma x_1^2+\gamma\alpha x_2^2+\alpha\beta x_3^2-\alpha(\gamma+\beta)x_2x_3-\beta(\alpha+\gamma)x_3x_1-\gamma(\beta+\alpha)x_1x_2=0.$$
It is the eleven-point conic of triangle $ABC$ with regard to $I$ and $J$ (see 1.6): it is the conic through the points $M_A(0,1,1)$, $M_B(1,0,1)$, $M_C(1,1,0)$, $AP\cap BC=H'_A(0,\beta,\gamma)$, $BP\cap CA=H'_B(\alpha,0,\gamma)$, $CP\cap AB=H'_C(\alpha,\beta,0)$, $A'(2\alpha+\beta+\gamma,\beta,\gamma)$, $B'(\alpha,\alpha+2\beta+\gamma,\gamma)$, $C'(\alpha,\beta,\alpha+\beta+2\gamma)$, $I$, and $J$. The pole of the line $\ell$ with regard to this conic is the point $Y'(2\alpha+\beta+\gamma, \alpha+2\beta+\gamma, \alpha+\beta+2\gamma)$, which is also a point of the line $PZ$. We denote this conic by $(Y')$.

Here is an alternative formulation of the main theorem.

**Theorem.** *Let $ABC$ be a triangle in the complex projective plane $\mathcal{P}$, $\ell$ be a line not through a vertex, and $I$ and $J$ be any two (different) points of $\ell$ not on a side* *of the triangle. Denote by $Q$ the pole of $\ell$ with respect to one of the four conics through $I$ and $J$ that are tangent to the sides of the triangle. If $Y$, $Y_A$, $Y_B$, and $Y_C$ are the poles of $\ell$ with respect to the conics determined by $I$, $J$, and the triples $ABC$, $QBC$, $QCA$, and $QAB$ respectively, while $Y'$, $Y'_A$, $Y'_B$, and $Y'_C$ are the respective poles with respect to their eleven-point conics with regard to $I$ and $J$, then $YY'$, $Y_AY'_A$, $Y_BY'_B$, and $Y_CY'_C$ concur at a point $S$.*

## 3. The Euclidean case

In this section we give applications of the main theorem in the Euclidean plane $\Pi$. Throughout the following sections, we only consider a general real triangle $ABC$ in $\Pi$, i.e., the side-lengths $a$, $b$, and $c$ are distinct and the triangle has no right angle.

**Corollary 1.** *Let $ABC$ be a triangle in $\Pi$ and assume that $\ell$ is the line at infinity of $\Pi$. Suppose that $P$ coincides with the orthocenter $H$ of $ABC$; then the conics of the pencil $\mathcal{B}(A,B,C,H)$ are rectangular hyperbolas and the involution on $\ell$, determined by $H$ (see 1.4), becomes the absolute (or orthogonal) involution with fixed points the cyclic points (or circle points) $J$ and $J'$ of $\Pi$. The four conics through $J, J'$ and tangent to the sidelines of $ABC$ are now the incircle and the excircles of $ABC$, and the points $Q = Q_1$, $Q_2$, $Q_3$, $Q_4$ become the incenter $I$, and the excenters $I_A$ (the line $II_A$ contains $A$), $I_B$, and $I_C$, respectively.*

*Next, the points $Z$, $Z_A$, $Z_B$, and $Z_C$, are the centroids of $ABC$, $IBC$, $ICA$, and of $IAB$ respectively. Finally, $P_A$, $P_B$, $P_C$ are the orthocenters $H_A$, $H_B$, $H_C$ of $IBC$, $ICA$, and $IAB$ respectively. Then the lines $HZ$, $H_AZ_A$, $H_BZ_B$, and $H_CZ_C$ concur at a point $S_H$.*

Remark that $HZ$, $H_AZ_A$, $H_BZ_B$, and $H_CZ_C$ are the Euler lines of the triangles $ABC$, $IBC$, $ICA$, and $IAB$, respectively. The point of concurrence of these Euler lines is known as the Schiffler point $S$ ([9]), but we prefer in this paper the notation $S_H$, since it results from setting $P = H$.

In connection with Remarks 4 and 5 of the foregoing section, and again working with $\ell$ as the line at infinity and $J$, $J'$ the cyclic points, the conic $(Y)$ becomes the circumcircle $(O)$ of $ABC$, $(Y')$ becomes its nine-point circle $(O')$, and $OO'$ is the Euler line.

In connection with Remark 5, we recall that the locus of the centers of the rectangular hyperbolas through $A$, $B$, $C$ (and $H$) is the nine-point circle $(O')$ of $ABC$ and that, for each point $U$ of the circumcircle $(O)$, the midpoint of $HU$ is a point of $(O')$ (and $O'$ is the midpoint of $HO$ on the Euler line).

The main theorem allows us to generalize the foregoing corollary as follows:

**Corollary 2.** *Let $ABC$ be a triangle and let $\ell$ be the line at infinity in $\Pi$. Choose a general point $P$ (*i.e.*, not on a sideline of $ABC$, not on $\ell$ and different from the centroid of $ABC$) and call $J$, $J'$ the tangent points on $\ell$ of the two conics of the pencil $\mathcal{B}(A,B,C,P)$ which are tangent to $\ell$ (these are the centers of the parabolas through $A, B, C$ and $P$). Denote by $Q$ the center of one of the four conics through $J$ and $J'$, which are tangent at the sidelines of $ABC$. Next, $Z$ is the centroid* *of $ABC$ and $Z_A, Z_B, Z_C$ are the centroids of the triangles $QBC$, $QCA$, $QAB$ respectively. Finally, $P_A$ ($P_B$, and $P_C$ respectively) is the fourth common point of the two parabolas through $Q$, $B$, $C$ (through $Q$, $C$, $A$, and through $Q$, $A$, $B$ respectively) and tangent to $\ell$ at $J$ and $J'$. Then the lines $PZ$, $P_AZ_A$, $P_BZ_B$, and $P_CZ_C$ concur at a point $S_P$.*

## 4. The use of trilinear coordinates

From now on, we work with trilinear coordinates $(x_1, x_2, x_3)$ with respect to the real triangle $ABC$ in the Euclidean plane $\Pi$ ([2, 5]): $A$, $B$, $C$, and the incenter $I$ of $ABC$, have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ respectively. The line at infinity $\ell$ has equation $ax_1 + bx_2 + cx_3 = 0$, where $a$, $b$, $c$ are the side-lengths of $ABC$. The orthocenter $H$, the centroid $Z$, the circumcenter $O$, and the center of the nine-point circle $O'$, have trilinear coordinates $\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$, $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(\cos A, \cos B, \cos C)$, and $(bc(a^2b^2 + a^2c^2 - (b^2 - c^2)^2), ca(b^2c^2 + b^2a^2 - (c^2 - a^2)^2), ab(c^2a^2 + c^2b^2 - (a^2 - b^2)^2))$ respectively. The equations of the circumcircle $(O)$ and the nine-point circle $(O')$ are $ax_2x_3 + bx_3x_1 + cx_1x_2 = 0$ and $x_1^2 \sin 2A + x_2^2 \sin 2B + x_3^2 \sin 2C - 2x_2x_3 \sin A - 2x_3x_1 \sin B - 2x_1x_2 \sin C = 0$.

The Schiffler point $S = S_H$ (the common point of the Euler lines of $ABC$, $IBC$, $ICA$, and $IAB$) has trilinear coordinates $\left(\frac{-a+b+c}{b+c}, \frac{a-b+c}{c+a}, \frac{a+b-c}{a+b}\right)$.

If $T$ is a point of $\Pi$, not on a sideline of $ABC$, reflect the line $AT$ about the line $AI$, and reflect $BT$ and $CT$ about the corresponding bisectors $BI$ and $CI$. The three reflections concur in the isogonal conjugate $T^{-1}$ of $T$, and $T^{-1}$ has trilinear coordinates $(t_2t_3, t_3t_1, t_1t_2)$ or $\left(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}\right)$ if $T$ has trilinear coordinates $(t_1, t_2, t_3)$. Examples: the circumcenter $O$ is the isogonal conjugate of the orthocenter $H$, and the centroid $Z$ is the isogonal conjugate of the Lemoine point (or symmedian point) $K(a, b, c)$.

Let us now interpret the main theorem (or Corollary 2) in the Euclidean case using trilinear coordinates, with $\ell : ax_1 + bx_2 + cx_3 = 0$ as line at infinity and with $P(\alpha, \beta, \gamma)$ a general point of $\Pi$. In fact, the only thing that we have to do, is to replace in the proof of the main theorem the equation $x_1 + x_2 + x_3 = 0$ of $\ell$, by $ax_1 + bx_2 + cx_3 = 0$, and a straightforward calculation gives us the following trilinear coordinates for the point $Q$: $(\sqrt{bc\alpha(b\beta + c\gamma)}, \sqrt{ca\beta(c\gamma + a\alpha)}, \sqrt{ab\gamma(a\alpha + b\beta)}) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$. Next, the points $Z$, $Z_A$, $Z_B$, and $Z_C$ are the centroids of $ABC$, $QBC$, $QCA$ and $QAB$ with trilinear coordinates $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(bc\mathcal{A}, c(a\mathcal{A}+2b\mathcal{B}+c\mathcal{C}), b(a\mathcal{A}+b\mathcal{B}+2c\mathcal{C}))$, $(c(2a\mathcal{A}+b\mathcal{B}+c\mathcal{C}), ca\mathcal{B}, a(a\mathcal{A}+b\mathcal{B}+2c\mathcal{C}))$, $(b(2a\mathcal{A}+b\mathcal{B}+c\mathcal{C}), a(a\mathcal{A}+2b\mathcal{B}+c\mathcal{C}), ba\mathcal{C})$, respectively. Now, for the points $P_A, P_B, P_C$, again after a straightforward calculation, we find the coordinates: $P_A(bc\mathcal{A}, c(c\mathcal{C} - a\mathcal{A}), b(b\mathcal{B} - a\mathcal{A}))$, $P_B(c(c\mathcal{C} - b\mathcal{B}), ca\mathcal{B}, a(a\mathcal{A} - b\mathcal{B}))$ and $P_C(b(b\mathcal{B} - c\mathcal{C}), a(a\mathcal{A} - c\mathcal{C}), ab\mathcal{C})$.

And finally, we find the trilinear coordinates of the point $S_P$, corresponding to $Q$:

$$\left(\frac{\mathcal{A}(-a\mathcal{A} + b\mathcal{B} + c\mathcal{C})}{b\mathcal{B} + c\mathcal{C}}, \frac{\mathcal{B}(a\mathcal{A} - b\mathcal{B} + c\mathcal{C})}{c\mathcal{C} + a\mathcal{A}}, \frac{\mathcal{C}(a\mathcal{A} + b\mathcal{B} - c\mathcal{C})}{a\mathcal{A} + b\mathcal{B}}\right).$$

Remark that we find for the case $P(\alpha,\beta,\gamma) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C})$:

$$\begin{aligned}\mathcal{A} &= \sqrt{bc\alpha(b\beta+c\gamma)} = \sqrt{\tfrac{bc}{\cos A}(\tfrac{b}{\cos B}+\tfrac{c}{\cos C})} = \sqrt{\tfrac{bc(b\cos C+c\cos B)}{\cos A\cos B\cos C}} \\ &= \sqrt{\tfrac{abc}{\cos A\cos B\cos C}} = \mathcal{B} = \mathcal{C}\end{aligned}$$

and $Q(\mathcal{A},\mathcal{B},\mathcal{C}) = I(1,1,1)$, while since $\mathcal{A} = \mathcal{B} = \mathcal{C}$, we get for $S_H$ the coordinates $\left(\frac{-a+b+c}{b+c}, \frac{a-b+c}{c+a}, \frac{a+b-c}{a+b}\right)$, which gives us the Schiffler point $S$.

Let us also calculate the trilinear coordinates of the points $Y$ and $Y'$, defined above as the centers of the conic $(Y)$ through $A$, $B$, $C$, $J$ and $J'$, and of the conic $(Y')$ through the midpoints of the sides of $ABC$ and through $J$, $J'$ (or the eleven-point conic of $ABC$ with regard to $J$ and $J'$; remark that $J$ and $J'$ are the cyclic points only when $P = H$):

$(Y)$ has equation $\alpha(b\beta+c\gamma)x_2x_3 + \beta(c\gamma+a\alpha)x_3x_1 + \gamma(a\alpha+b\beta)x_1x_2 = 0$ and center $Y(bc(b\beta+c\gamma), ca(c\gamma+a\alpha), ab(a\alpha+b\beta))$,

$(Y')$ has equation $a\beta\gamma x_1^2 + b\gamma\alpha x_2^2 + c\alpha\beta x_3^2 - \alpha(\gamma c+b\beta)x_2x_3 - \beta(a\alpha+c\gamma)x_3x_1 - \gamma(b\beta+a\alpha)x_1x_2 = 0$ and center $Y'(bc(2a\alpha+b\beta+c\gamma), ca(a\alpha+2b\beta+c\gamma), ab(\alpha+b\beta+2c\gamma))$.

Remark that $Q = \sqrt{P*Y}$, with the notation $\sqrt{(x_1,x_2,x_3)*(y_1,y_2,y_3)} = (\sqrt{x_1y_1}, \sqrt{x_2y_2}, \sqrt{x_3y_3})$.

Recall that the coordinate transformation between trilinear coordinates $(x_1,x_2,x_3)$ with regard to $\triangle ABC$ and trilinear coordinates $(x_1',x_2',x_3')$ with regard to the medial triangle $M_AM_BM_C$, is given by ([5, p.207]):

$$\begin{pmatrix} ax_1 \\ bx_2 \\ cx_3 \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ a & 0 & c \\ a & b & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}.$$

Now, this gives for $(x_1,x_2,x_3)$ the coordinates of the point $Y$, if $(x_1',x_2',x_3')$ are the coordinates $(\alpha,\beta,\gamma)$ of $P$ and it gives for $(x_1,x_2,x_3)$ the coordinates of $Y'$ if $(x_1',x_2',x_3')$ are the coordinates of $Y$. Moreover, $\triangle ABC$ and its medial triangle are homothetic. As a corollary, we have that if $P$ ($Y$, respectively) is *triangle center* $X(k)$ for $\triangle ABC$ (for the definition of triangle center, see [5, p.46]), then $Y$ ($Y'$ respectively) is center $X(k)$ for $\triangle M_AM_BM_C$.

## 5. Applications

In this section we choose $P(\alpha,\beta,\gamma)$ as a triangle center of the triangle $ABC$ and calculate the coordinates of the corresponding points $Y$, $Y'$, $Q$ and $S_P$ (sometimes $Y'$ and $S_P$ are not given).

Remark that $P$ must be different from the centroid $Z$ of $ABC$. The triangle centers are taken from Kimberling's list : $X(1)$, $X(2)$, $\ldots$, $X(2445)$ (list until 29 March 2004, see [6]). When we found the points $Y$, $Y'$, $Q$ or $S_P$ in this list, we give the number $X(\cdots)$ and if possible, the name of the center. But, without doubt, we overlooked some centers and more points $Y$, $Y'$, $Q$, $S_P$ than indicated will occur in Kimberling's list. Several times, only the first trilinear coordinate is given: the second and the third are obtained by cyclic permutations.

5.1. The first example is of course:
$P(\alpha,\beta,\gamma) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}) = X(4)$ (orthocenter),
$Y = O(\cos A, \cos B, \cos C) = X(3)$ (circumcenter),
$Y' = O'(bc(a^2b^2 + a^2c^2 - (b^2 - c^2)^2), \cdots, \cdots) = X(5)$ (nine-point center),
$Q = I(1, 1, 1) = X(1)$ (incenter), and
$S_H = S\left(\frac{-a+b+c}{b+c}, \cdots, \cdots\right) = X(21)$ (Schiffler point).

5.2. $P(\alpha,\beta,\gamma) = I(1, 1, 1) = X(1)$,
$Y = (\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}) = X(10)$ (Spieker point = incenter of the medial triangle $M_AM_BM_C$),
$Y' = (\frac{2a+b+c}{a}, \cdots, \cdots) = X(1125)$ (Spieker point of the medial triangle),
$Q = (\sqrt{bc(b + c)}, \cdots, \cdots)$, and
$S_I = (\sqrt{bc(b + c)}\frac{-a\sqrt{bc(b+c)}+b\sqrt{ca(c+a)}+c\sqrt{ab(a+b)}}{b\sqrt{ca(c+a)}+c\sqrt{ab(a+b)}}, \cdots, \cdots)$.

5.3. $P(\alpha,\beta,\gamma) = K(a, b, c) = X(6)$ (Lemoine point),
$Y = (\frac{b^2+c^2}{a}, \cdots, \cdots) = X(141)$ = Lemoine point of medial triangle,
$Y' = (\frac{2a^2+b^2+c^2}{a}, \cdots, \cdots)$,
$Q = (\sqrt{b^2 + c^2}, \cdots, \cdots)$, and
$S_K = (\sqrt{b^2 + c^2}\frac{-a\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}}{b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}}, \cdots, \cdots)$.

5.4. $P(\alpha,\beta,\gamma) = (\frac{1}{a(-a+b+c)}, \cdots, \cdots) = X(7)$ (Gergonne point),
$Y = (-a + b + c, a - b + c, a + b - c) = X(9)$ (Mittenpunkt = Lemoine point of the excentral triangle $I_AI_BI_C$ = Gergonne point of medial triangle),
$Y' = (bc(a(b+c)-(b-c)^2), \cdots, \cdots) = X(142)$ (Mittenpunkt of medial triangle),
$Q = (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}) = X(366)$, and
$S_{X(7)} = (\frac{1}{\sqrt{a}}\frac{-\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{b}+\sqrt{c}}, \cdots, \cdots)$.

5.5. $P(\alpha,\beta,\gamma) = (\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}) = X(100)$,
$Y = (bc(b - c)^2(-a + b + c), \cdots, \cdots) = X(11)$ (Feuerbach point = $X(100)$ of medial triangle),
$Y' = (bc((a - b)^2(a + b - c) + (c - a)^2(a - b + c)), \cdots, \cdots)$ (Feuerbach point of medial triangle), and $Q = (\sqrt{bc(b - c)(-a + b + c)}, \cdots, \cdots)$.

In the foregoing examples, the coordinates of the point $S_P$ are mostly rather complicated. Another method is to start with the coordinates of the point $Q$: if $(k, l, m)$ are the trilinear coordinates of $Q$, then a short calculation shows that it corresponds with the point $P(\frac{1}{a(-a^2k^2+b^2l^2+c^2m^2)}, \cdots, \cdots)$ and $S_P$ becomes the point $(k\frac{-ak+bl+cm}{bl+cm}, \cdots, \cdots)$. Finally, the coordinates of $Y$ and $Y'$ are $(ak^2(-a^2k^2 + b^2l^2 + c^2m^2), \cdots, \cdots)$, and $(bc(a^2k^2(b^2l^2 + c^2m^2) - (b^2l^2 - c^2m^2)^2), \cdots, \cdots)$, respectively. Here are some examples.

5.6. $Q(k,l,m) = K(a,b,c) = X(6)$(Lemoine point),
$P = (\frac{1}{a(-a^4+b^4+c^4)}, \cdots, \cdots) = X(66) = X(22)^{-1}$ ($X(22)$ is the Exeter point),
$Y = (a^3(-a^4+b^4+c^4), \cdots, \cdots) = X(206)$ ($X(66)$ of medial triangle),
$Y' = (bc(a^4(b^4+c^4)-(b^4-c^4)^2), \cdots, \cdots)$ ($X(206)$ of medial triangle), and
$S_{X(66)} = (\frac{a(-a^2+b^2+c^2)}{b^2+c^2}, \cdots, \cdots) = (\frac{\cos A}{b^2+c^2}, \cdots, \cdots) = X(1176)$.

5.7. $Q(k,l,m) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}) = X(4)$ (orthocenter),
$P = (\frac{1}{a(-\frac{a^2}{\cos^2 A}+\frac{b^2}{\cos^2 B}+\frac{c^2}{\cos^2 C})}, \cdots, \cdots)$,
$Y = (\frac{a}{\cos^2 A}(-\frac{a^2}{\cos^2 A}+\frac{b^2}{\cos^2 B}+\frac{c^2}{\cos^2 C}), \cdots, \cdots)$, and
$S_P = (\frac{\cos A - \cos B \cos C}{\cos^2 A}, \cdots, \cdots)$.

5.8. $Q(k,l,m) = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point),
$P = (\frac{1}{a(-(b+c)^2+(c+a)^2+(a+b)^2)}, \cdots, \cdots) = X(596)$,
$Y = \left(\frac{(b+c)^2}{a}(-(b+c)^2+(c+a)^2+(a+b)^2), \cdots, \cdots\right)$ ($X(596)$ of medial triangle), and
$S_P = (\frac{b+c}{2a+b+c}, \frac{c+a}{a+2b+c}, \frac{a+b}{a+b+2c})$.

We also can start with the coordinates of the point $Y(p,q,r)$, then
$P = (\frac{-ap+bq+cr}{a}, \cdots, \cdots)$,
$Y'(bc(bq+cr), \cdots, \cdots)$, and
$Q = \sqrt{P * Y} = (\sqrt{\frac{p(-ap+bq+cr)}{a}}, \cdots, \cdots)$. Here are some examples.

5.9. $Y(p,q,r) = I(1,1,1) = X(1)$, $P = (\frac{-a+b+c}{a}, \frac{a-b+c}{b}, \frac{a+b-c}{c}) = X(8)$ (Nagel point),
$Y' = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point = incenter of medial triangle),
$Q = \left(\sqrt{\frac{-a+b+c}{a}}, \cdots, \cdots\right) = X(188)$, and
$S_P = (\mathcal{A}\frac{-a\mathcal{A}+b\mathcal{B}+c\mathcal{C}}{b\mathcal{B}+c\mathcal{C}}, \cdots, \cdots)$ with $Q(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

5.10. $Y = K(a,b,c) = X(6)$ (Lemoine point),
$P = (\frac{-a^2+b^2+c^2}{a}, \cdots, \cdots) = (\frac{cosA}{a^2}, \cdots, \cdots) = X(69)$,
$Y' = (\frac{b^2+c^2}{a}, \cdots, \cdots) = X(141)$ (Lemoine point of medial triangle), and
$Q = (\sqrt{-a^2+b^2+c^2}, \cdots, \cdots)$.

5.11. $Y = (\frac{2a+b+c}{a}, \cdots, \cdots) = X(1125)$ (Spieker point of medial triangle),
$P = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point),
$Y' = (\frac{2a+3b+3c}{a}, \cdots, \cdots)$ ($X(1125)$ of medial triangle), and
$Q = (bc\sqrt{(b+c)(2a+b+c)}, \cdots, \cdots)$.

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