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Minimal Chords in Angular Regions

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Abstract. We use synthetic geometry to show that in an angular region minimal chords having a prescribed direction form a ray which is constructible with ruler and compass.

Let P be a fixed point inside a circle of center O. It is well-known that among the chords containing P one of minimal length is perpendicular to the diameter through P, if $P \neq O$, or is any diameter, if $P = O$. Consequently, such a chord is always constructible with ruler and compass.

When it comes to geometrically constructing minimal chords through given points in convex regions the circle is in some sense a singular case. Indeed, as shown in [1] this task is impossible even in the case of the conics. However, in general *it is* possible to construct all the points inside a convex region which support minimal chords parallel to a given direction. We proved this in [1, 2] by analytical means, with special emphasis on the conics.

The purpose of this note is to prove the same thing for angular regions, via essentially a purely geometrical argument.

To this end let $\angle AOB$ be an angle of vertex O and sides $\overrightarrow{OA}$, $\overrightarrow{OB}$, such that O, A, and B are not colinear, and let P be a point inside the angle. By definition, a *chord* in this angle is a straight segment $\overline{MN}$ such that $M \in \overrightarrow{OA}$ and $N \in \overrightarrow{OB}$. A continuity argument makes clear that among the chords containing P there is at least one of minimal length, that is a minimal chord through P in the given angle.

Problem. *Given a direction in the plane of $\angle AOB$, construct with ruler and compass the geometric locus of all the points inside the angle which support minimal chords parallel to that direction.*

In order to solve this problem we need the following

Lemma. *Inside $\angle AOB$ consider the chord $\overline{MN}$, $M \in \overrightarrow{OA}$, $N \in \overrightarrow{OB}$, such that $\angle OMN$ and $\angle ONM$ are acute angles. If P is the foot of the perpendicular on $\overline{MN}$ through the point Q diametrically opposite O on the circle circumscribed about $\triangle OMN$, then $\overline{MN}$ is the unique minimal chord through P inside $\angle AOB$. P is seen to be the unique point inside $\overline{MN}$ such that $\overline{ML} \cong \overline{NP}$, where L is*

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the foot of the perpendicular from O on $\overline{MN}$. Moreover, any point on the ray $\overrightarrow{OP}$ supports an unique minimal chord, parallel to $\overline{MN}$.

Proof. Clearly, Q is an interior point to $\angle AOB$, situated on the other side of the line $\overleftrightarrow{MN}$ with respect to O, and $\overline{MQ} \perp \overrightarrow{OA}$ and $\overline{NQ} \perp \overrightarrow{OB}$. Since $\angle OMN$ and $\angle ONM$ are acute angles, and $\angle QMN$ and $\angle QNM$ are acute angles too, as complements of acute angles , the points P and L described in the statement of the Lemma are interior points to the segment $\overline{MN}$. (See Figure 1).

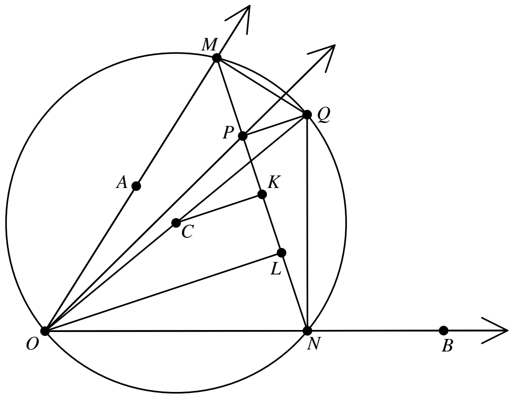


Figure 1

Let us prove first that $\overline{MN}$ is a minimal chord through P in $\angle AOB$. Let $\overline{M'N'}$, $M' \in \overrightarrow{OA}$, $N' \in \overrightarrow{OB}$, $P \in \overline{M'N'}$, be another chord through P (See Figure 2).

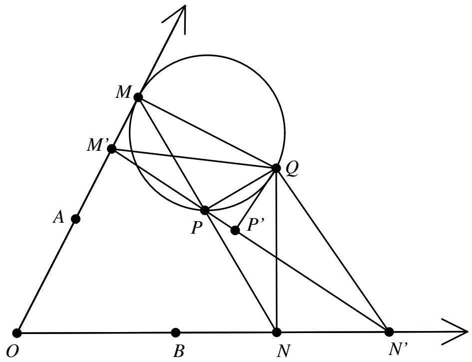


Figure 2

Notice now that the following angle inequalities hold:

$$\angle QM'P < \angle QMP, \;\; \angle QN'P < \angle QNP \tag{1}$$

Indeed, since the circle circumscribed about $\triangle MPQ$ is tangent to the ray $\overrightarrow{OA}$ at M, the point M' is located outside this circle. Now $\angle QMP$ and $\angle QM'P$ are precisely the angles the segment $\overline{PQ}$ is seen from M, respectively M'. Since M belongs to the circle circumscribed about $\triangle MPQ$ and M' is outside this circle, the inequality $\angle QM'P < \angle QMP$ becomes obvious. The other inequality (1) can be proved in a similar fashion.

The inequalities (1) prove that $\angle QM'N'$ and $\angle QN'M'$ are acute angles too, thus the foot P' of the perpendicular from Q on the line $\overleftrightarrow{M'N'}$ belongs to the interior of the segment $\overline{M'N'}$.

Notice now that

$$MQ < M'Q, \;\; NQ < N'Q, \;\; P'Q < PQ.$$

The above inequalities are obvious since in a right triangle a leg is shorter than the hypothenuse. Consequently, the Pythagorean Theorem yields

$$MP = \sqrt{MQ^2 - PQ^2} < \sqrt{M'Q^2 - P'Q^2} = M'P',$$

and similarly, $NP < N'P'$. In conclusion,

$$MN = MP + NP < M'P' + N'P' = M'N',$$

and so $\overline{MN}$ is indeed the unique minimal chord through P in $\angle AOB$.

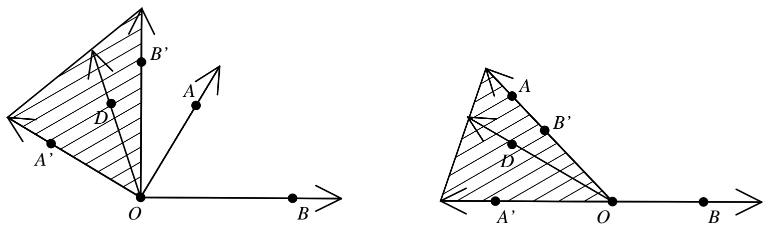
The perpendicular line on $\overline{MN}$ through the center C of the circle circumscribed about the quadrilateral $OMQN$ intersects $\overline{MN}$ at its midpoint K (See Figure 1). Clearly, $\overline{KP} \cong \overline{KL}$, and so $\overline{ML} \cong \overline{NP}$ as stated.

Finally, the fact that any point on the ray $\overrightarrow{OP}$ supports an unique minimal chord parallel to $\overline{MN}$ is an immediate consequence of standard properties of similar triangles in the context of what was proved above. □

To $\angle AOB$ we associate now another angle, $\angle A'OB'$, according to the following recipe:

$a)$ If $\angle AOB$ is *acute* then $\angle A'OB'$ is obtained by rotating $\angle AOB$ counter-clockwise $90°$ around O.

$b)$ If $\angle AOB$ is *not acute* (so it is either right or obtuse) then $\angle A'OB'$ is the supplementary angle to $\angle AOB$ along the line $\overleftrightarrow{OB}$ (See Figure 3).



a) Acute *b) Not Acute*

Figure 3

Definition. A ray $\overrightarrow{OD}$ is called an *admissible direction* for $\angle AOB$ if D is a point interior to $\angle A'OB'$.

It is easy to see that $\overrightarrow{OD}$ is an admissible direction for $\angle AOB$ if and only if any parallel line to $\overrightarrow{OD}$ through a point interior to $\angle AOB$ determines a chord $\overline{MN}$ such that $\angle OMN$ and $\angle ONM$ are acute angles.

Theorem. *Any point P inside $\angle AOB$ supports an unique minimal chord, parallel to an admissible direction. The geometric locus of all the points inside $\angle AOB$ which support minimal chords parallel to a given admissible direction can be constructed with ruler and compass as follows:*

$i)$ *Construct first the line $\overleftrightarrow{OL}$ perpendicular to the admissible direction, the point L being interior to $\angle AOB$.*

$ii)$ *Construct next the perpendicular through L to the line $\overleftrightarrow{OL}$, which intersects $\overrightarrow{OA}$ at M and $\overrightarrow{OB}$ at N.*

$iii)$ *Inside the segment $\overline{MN}$ construct the point P such that $\overline{NP} \cong \overline{ML}$.*

$iv)$ *Finally, construct the ray $\overrightarrow{OP}$, which is the desired geometric locus.*

Using the Lemma, an alternative construction can be provided by using the circle circumscribed about $\triangle OMN$, where the point M is chosen arbitrarily on $\overrightarrow{OA}$ and $N \in \overrightarrow{OB}$ is such that $\overline{MN}$ is parallel to the given admissible direction.

Proof. Let P be a fixed point inside $\angle AOB$. The proof splits naturally into two cases, according to $\angle AOB$ being acute or not.

$a)$ *$\angle AOB$ is acute*. Let $\overline{M_1N_1}$ be the perpendicular segment through P to $\overrightarrow{OA}$, $M_1 \in \overrightarrow{OA}$, $N_1 \in \overrightarrow{OB}$ and let $\overline{M_2N_2}$ be the perpendicular segment through P to $\overrightarrow{OB}$, $M_2 \in \overrightarrow{OA}$, $N_2 \in \overrightarrow{OB}$. Define now a function $f : \overline{M_1M_2} \longrightarrow \mathbf{R}$, by

$$f(M) = ML - NP, \quad M \in \overline{M_1M_2}, \tag{2}$$

where N is the intersection point of the line $\overleftrightarrow{MP}$ with $\overrightarrow{OB}$, and L is the foot of the perpendicular from O to the segment $\overline{MN}$ (See Figure 4).

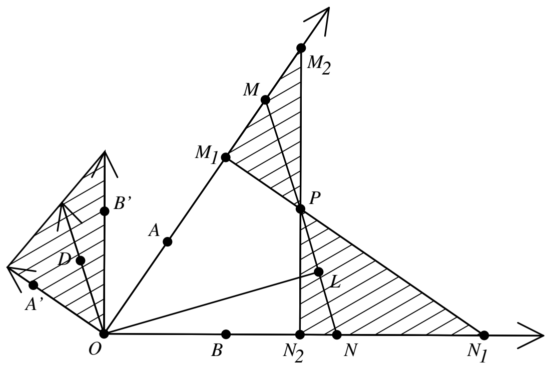


Figure 4

Clearly, this is a continuous function and $f(M_1) = -N_1P < 0$ and $f(M_2) = M_2P > 0$. By the intermediate value property there is some point $M \in \overline{M_1M_2}$ such that $f(M) = 0$, or equivalently $\overline{NP} \cong \overline{ML}$. According to the above Lemma, for this point M the chord $\overline{MN}$ is the unique minimal chord through P. It is also obvious that $\overline{MN}$ is parallel to an admissible direction.

$b)$ $\angle AOB$ *is not acute*. The proof in this case is a variant of that given at $a)$. Let M_0 be the point where the parallel line through P to $\overrightarrow{OB}$ intersects the ray $\overrightarrow{OA}$. Without loss of generality we can assume that M_0 is located between O and A. Defining now the function $f : \overrightarrow{M_0A} \longrightarrow \mathbf{R}$ by the same formula (2), we see that for points M close to M_0, $f(M)$ takes negative values and for points M far away on $\overrightarrow{M_0A}$, $f(M)$ takes positive values. One more time, the intermediate value property and the above Lemma guarantee the existence of an unique minimal chord through P, which is also parallel to an admissible direction.

Given now an admissible direction, the previous Lemma justifies the construction of the desired geometric locus as indicated in the statement of the theorem if we can prove that this locus does not contain points outside the ray $\overrightarrow{OP}$ described at $iv)$. Indeed this is the case since if there were other points then the equation $\overline{NP} \cong \overline{ML}$ would not hold. However, we have just proved that this equation is necessary for minimal chords. $\square$

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[2] N. Anghel, Geometric loci associated to certain minimal chords in convex regions, *J. Geom.*, 66 (1999) 1–16.

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