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Three Pairs of Congruent Circles in a Circle

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Abstract. Consider a closed chain of three pairs of congruent circles of radii a, b, c. The circle tangent internally to each of the 6 circles has radius $R = a + b + c$ if and only if there is a pair of congruent circles whose centers are on a diameter of the enclosing circle. Non-neighboring circles in the chain may overlap. Conditions for nonoverlapping are established. There can be a "central circle" tangent to four of the circles in the chain.

1. Introduction

Consider a closed chain of three pairs of congruent circles of radii a, b, c, as shown in Figure 1. Each of the circles is tangent internally to the enclosing circle (O) of radius R and tangent externally to its two neighboring circles.

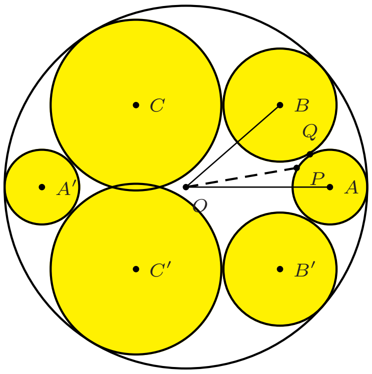

Figure 1A: $(abcacb)$

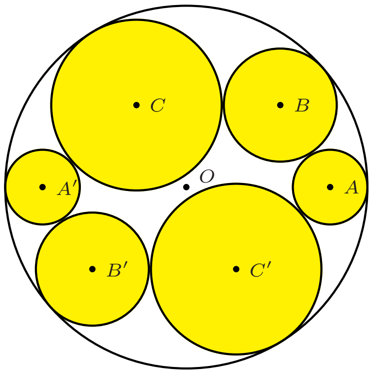

Figure 1B: $(abcabc)$

The essentially distinct arrangements, depending on the number of pairs of congruent neighboring circles, are

(A): $(aabcbc)$	(B): $(aacbbc)$
(C): $(aabbcc)$	(D): $(aaaabb)$
(E): $(abcabc)$, $(abcacb)$	(F): $(aabaab)$, $(aaabab)$
(G): $(aaaaaa)$	

Figures 1A and 1B illustrate the pattern (E). Patterns (D) and (F) have $c = a$. In pattern (G), $b = c = a$.

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According to [1, 3], in 1877 Sakuma proved $R = a + b + c$ for patterns (E). Hiroshi Okumura [1] published a much simpler proof. Unaware of this, Tien [4] rediscovered the theorem in 1995 and published a similar, simple proof. It is easy to see by symmetry that in each of the patterns (E), (F), (G), there is a pair of congruent circles with centers on a diameter of the enclosing circle. Let us call such a pair a *diametral* pair. Here is a stronger theorem:

Theorem 1. *In a closed chain of three pairs of congruent circles of radii a, b, c tangent internally to a circle of radius R, $R = a + b + c$ if and only if the closed chain contains a diametral pair of circles.*

In Figure 1, two non-neighboring circles intersect. The proof for $R = a + b + c$ does not forbid such an intersection. Sections 4 and 5 are about avoiding intersecting circles and about adding a "central" circle.

2. Preliminaries

In Figure 1, the enclosing circle (O) of radius R centers at O and the circles (A), (B), (C) of radii a, b, c, center at A, B, C, respectively. The circles (A'), (B'), (C') are also of radii a, b, c respectively.

Suppose two circles (A) and (B) of radii a and b are tangent externally each other, and each tangent internally to a circle $O(R)$. We denote the magnitude of angle AOB by θ_{ab}. See Figure 2A. This clearly depends on R. If $a < \frac{R}{2}$, then we can also speak of θ_{aa}. Note that the center O is outside each circle of radius a.

Lemma 2. (a) *If* $a < \frac{R}{2}$, $\sin \frac{\theta_{aa}}{2} = \frac{a}{R-a}$. (See Figure 2A).
(b) $\cos \theta_{bc} = \frac{(R-b)^2+(R-c)^2-(b+c)^2}{2(R-b)(R-c)}$. (See Figure 2B).

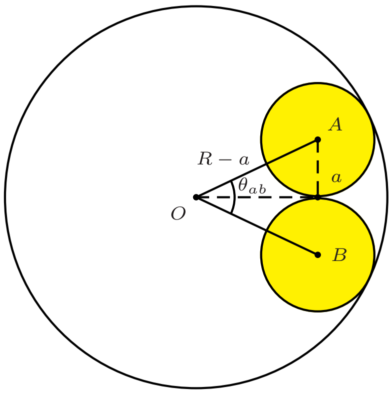

Figure 2A

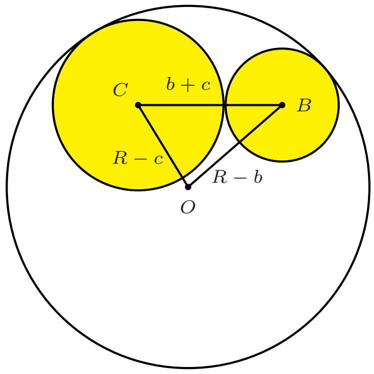

Figure 2B

Proof. These are clear from Figures 2A and 2B. □

Lemma 3. *If a and b are unequal and each $< \frac{R}{2}$, then $\theta_{aa} + \theta_{bb} > 2\theta_{ab}$.*

Proof. In Figure 1A, consider angle AOP, where P is a point on the circle (A). The angle AOP is maximum when line OP is tangent to the circle (A). This maximum is $\frac{\theta_{aa}}{2} \geq \angle AOQ$, where Q is the point of tangency of (A) and (B). Similarly, $\frac{\theta_{bb}}{2} \geq \angle BOQ$, and the result follows. $\square$

Corollary 4. *If a, b, c are not the same, then* $\theta_{aa} + \theta_{bb} + \theta_{cc} > \theta_{ab} + \theta_{bc} + \theta_{ca}$.

Proof. Write

$$\theta_{aa} + \theta_{bb} + \theta_{cc} = \frac{\theta_{aa} + \theta_{bb}}{2} + \frac{\theta_{bb} + \theta_{cc}}{2} + \frac{\theta_{cc} + \theta_{aa}}{2}$$

and apply Lemma 3. $\square$

3. Proof of Theorem 1

Sakuma, Okumura [1] and Tien [4] have proved the sufficiency part of the theorem. We need only the necessity part. This means showing that for distinct a, b, c in patterns (A) through (D) which do not have a diametral pair of circles, the assumption of $R = a + b + c$ causes contradictions. In patterns (E) with a pair of diametral circles and $R = a + b + c$, the sum of the angles around the center O of the enclosing circle is $2(\theta_{ab} + \theta_{bc} + \theta_{ca}) = 2\pi$, that is,

$$\theta_{ab} + \theta_{bc} + \theta_{ca} = \pi.$$

Pattern (A): $(aabcbc)$. The sum of the angles around O is

$$\begin{aligned} \theta_{aa} + \theta_{ab} + \theta_{bc} + \theta_{cb} + \theta_{bc} + \theta_{ca} =& \theta_{ab} + \theta_{bc} + \theta_{ca} + (\theta_{aa} + 2\theta_{bc}) \\ =& \pi + (\theta_{aa} + 2\theta_{bc}). \end{aligned}$$

This is 2π if and only if $(\theta_{aa} + 2\theta_{bc}) = \pi$, or $\frac{\pi}{2} - \frac{\theta_{aa}}{2} = \theta_{bc}$. The cosines of these angles, Lemma 2 and the assumption $R = a + b + c$ lead to

$$\frac{a}{b+c} = \frac{a^2 + ab + ac - bc}{(a+b)(a+c)},$$

which gives

$$(a-b)(a-c)(a+b+c) = 0,$$

an impossibility, if a, b, c are distinct.

Pattern (B): $(aacbbc)$. If $a > \frac{R}{2}$ or $b > \frac{R}{2}$, then the neighboring tangent circles of radii a or b, respectively, cannot fit inside the enclosing circle of radius $R = a + b + c$. For this equation to hold, it must be that $a \leq \frac{R}{2}$ and $b \leq \frac{R}{2}$. Then, O is outside $A(a)$ and $B(b)$. The sum of the angles around O exceeds 2π, by Lemma 3:

$$\begin{aligned} & \theta_{aa} + \theta_{ac} + \theta_{cb} + \theta_{bb} + \theta_{bc} + \theta_{ca} \\ =& (\theta_{aa} + \theta_{bb}) + 2(\theta_{bc} + \theta_{ca}) \\ >& 2(\theta_{ab} + \theta_{bc} + \theta_{ca}) \\ =& 2\pi. \end{aligned}$$

Patterns (C) and (D): $(aabbcc)$ *and* $(aaaabb)$. For $R = a + b + c$ to hold, O must be outside $A(a)$, $B(b)$, $C(c)$. Again, the sum of the angles around O exceeds 2π. For pattern (C),

$$\begin{aligned}
&\theta_{aa} + \theta_{ab} + \theta_{bb} + \theta_{bc} + \theta_{cc} + \theta_{ca} \\
=&(\theta_{aa} + \theta_{bb} + \theta_{cc}) + (\theta_{ab} + \theta_{bc} + \theta_{ca}) \\
>&(\theta_{ab} + \theta_{bc} + \theta_{ca}) + (\theta_{ab} + \theta_{bc} + \theta_{ca}) \\
=&2\pi.
\end{aligned}$$

Here, the inequality follows from Corollary 4 for a, b, c, not all the same.

For pattern (D) with $c = a$, the inequality remains true. This completes the proof of Theorem 1.

Remark. A narrower version of Theorem 1 treats a, b, c as variables, instead of any particular lengths. The proof for this version is simple. We see that when no pair of the enclosed circles is diametral, at least one pair has its two circles next to each other. Let these two be point circles and let the other four circles be of the same radius. Then the six circles become three equal tangent circles tangentially enclosed in a circle. In this special case $R = a + b + c = 0 + a + a$ is false. Then, a, b, c cannot be variables.

4. Nonoverlapping arrangements

Patterns (A) through (G) are adaptable to hands-on activities of trying to fit chains of three pairs of congruent circles into an enclosing circle of a fixed radius R. Most of the essential patterns have inessential variations. Assuming $a \le b \le c$, patterns (E) have four variations:

$$\begin{aligned}
\mathrm{E}_1 :&\quad (abcabc) \\
\mathrm{E}_2 :&\quad (cabcba) \\
\mathrm{E}_3 :&\quad (abcacb) \\
\mathrm{E}_4 :&\quad (bcabac)
\end{aligned}$$

For hands-on activities, it is desirable to find the conditions for the enclosed circles in patterns (E) not to overlap. We find the bounds of the ratio $\frac{a}{R}$ in these patterns.

4.1. *Patterns* E_1 *and* E_2. The largest circles (C) and $C')$ are diametral. For a nonoverlapping arrangement, Clearly, $a \le \frac{1}{3}R$ and $c \le \frac{1}{2}R$.

In Figure 3, a circle of radius b' is tangent externally to the two diametral circles of radii c, and internally to the enclosing circle of radius R. From

$$(b' + c)^2 = (R - b')^2 + (R - c)^2,$$

we have $b' = \frac{R(R-c)}{R+c}$. It follows that in a nonoverlapping patterns E_1 and E_2, with $\frac{1}{3}R \le c \le \frac{1}{2}R$, we have

$$b + c \le b' + c = \frac{R^2 + c^2}{R + c} \le \frac{5}{6}R.$$

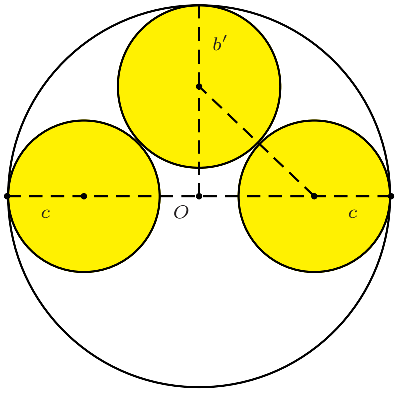

Figure 3

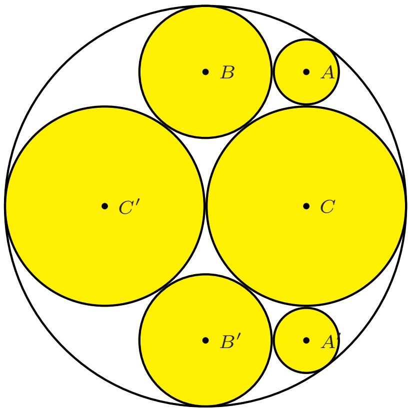

Figure 4

From this, $a \geq \frac{1}{6}R$. Figure 4 shows a nonoverlapping arrangement with $a = \frac{1}{6}R$, $b = \frac{1}{3}R$, $c = \frac{1}{2}R$. It is clear that for every a satisfying $\frac{1}{6}R \leq a \leq \frac{1}{3}R$, there are nonoverlapping patterns E_1 and E_2 (with $a \leq b \leq c$).

4.2. *Patterns* E_3 *and* E_4. In these cases the largest circles (C) and (C') are not diametral.

Lemma 5. *If three circles of radii x, z, z are tangent externally to each other, and are each tangent internally to a circle of radius R, then*

$$z = \frac{4Rx(R-x)}{(R+x)^2}.$$

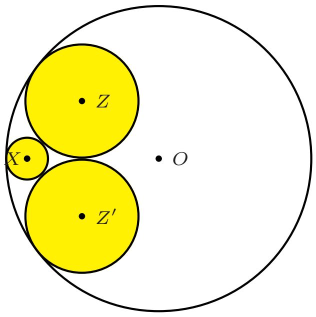

Figure 5

Proof. By the Descartes circle theorem [2], we have

$$2\left(\frac{1}{R^2} + \frac{1}{x^2} + \frac{2}{z^2}\right) = \left(-\frac{1}{R} + \frac{1}{x} + \frac{2}{z}\right)^2,$$

from which the result follows. □

Theorem 6. *For a given R, a nonoverlapping arrangement of pattern $E_3(abcacb)$ or $E_4(bcabac)$ with $a \le b \le c$ and $a+b+c = R$ exists if $\gamma R \le a \le \frac{1}{3}R$, where*

$$\gamma = \frac{1+\sqrt[3]{19+12\sqrt{87}}+\sqrt[3]{19-12\sqrt{87}}}{6} \approx 0.25805587\cdots.$$

Proof. For $b = a$ and the largest $c = R - 2a$ for a nonoverlapping arrangement $\mathrm{E}_3(abcacb)$, Lemma 5 gives

$$\frac{4Ra(R-a)}{(R+a)^2} - (R-2a) = \frac{f(\frac{a}{R})R^3}{(R+a)^2} = 0,$$

where $f(x) = 2x^3 - x^2 + 4x - 1$. It has a unique real root γ given above.

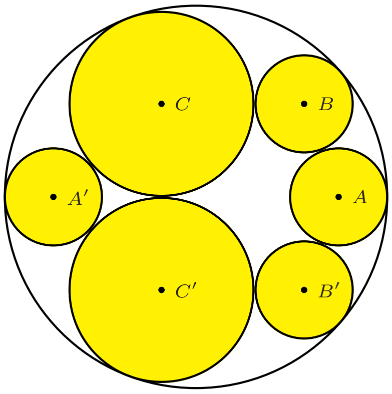

Figure 6

Figure 6 shows a nonoverlapping arrangement E_3 with $a = b = \gamma R$, and $c = (1-2\gamma)R$. For $\gamma R \le a \le \frac{1}{3}R$, from the figure we see that (C) and (C') and the other circles cannot overlap in arrangements of patterns $\mathrm{E}_3(abcacb)$ and $\mathrm{E}_4(bcabac)$. $\square$

Corollary 7. *The sufficient condition $\gamma R \le a \le \frac{1}{3}R$ also applies to patterns E_1 and E_2.*

Outside the range $\gamma R \le a \le \frac{R}{3}$, patterns $\mathrm{E}_3(abcacb)$ and $\mathrm{E}_4(bcabac)$ still can have nonoverlapping circles. Both of the patterns involve Figure 5 and $z = \frac{4Rx(R-x)}{(R+x)^2}$, with $z = c$, $x = a$ or b, and $a \le b \le c$.

The equation gives the smallest $x = a = (3-2\sqrt{2})R \approx 0.1715\cdots R$ corresponding to the largest $b = c = (\sqrt{2}-1)R \approx 0.4142\cdots R$ and the largest $x = b = \frac{R}{3}$ corresponding to the largest $c = \frac{R}{2}$. Thus, the nonoverlapping conditions are $(3-2\sqrt{2})R \le x \le \frac{R}{3}$ and $c \le \frac{4Rx(R-x)}{(R+x)^2}$.

For $x \ge \frac{R}{3}$, circles (Z) and (Z') overlap with (X'), which is diametral with (X). Now Figure 3 and the associated $b' = \frac{R(R-c)}{R+c}$ are relevant. With b' replaced by c and c by b, the equation becomes $c = \frac{R(R-b)}{R+b}$. By this equation, when b varies from $\frac{R}{3}$ to $(\sqrt{2}-1)R$, $c \geq b$ varies from $\frac{R}{2}$ to $(\sqrt{2}-1)R$. Thus, the nonoverlapping conditions are $\frac{R}{3} \leq b \leq (\sqrt{2}-1)R$ and $c \leq \frac{R(R-b)}{R+b}$. The case of $b > (\sqrt{2}-1)R$ makes $b > c$ and the largest pair of circles diametral, already covered in §4.1.

5. The central circle and avoiding intersecting circles

Obviously, pattern (G) $(aaaaaa)$ admits a "central" circle tangent to all 6 circles of radii a. In patterns (F) $(aabaab)$, (aaabab), we can add a central circle tangent to the four circles of radius a. Figure 7 shows the less obvious central circle for $(abcacb)$ of pattern (E).

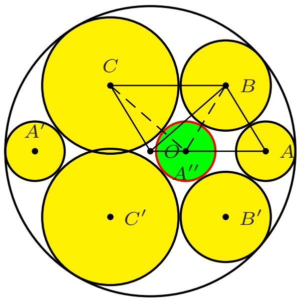

Figure 7

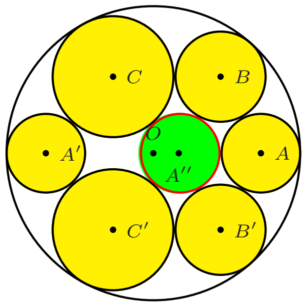

Figure 8

Theorem 8. *Consider a closed chain of pattern* $(abcacb)$. *There is a "central" circle of radius* a *tangent to the four circles of radii* b *and* c. *This circle does not overlap with the circle* $A(a)$ *if*

$$a \leq \frac{b(b+c)}{2c},$$

where $b \leq c$.

Proof. In Figure 7, the pattern of the chain tells that $R = a + b + c$. The central circle centered at A'' has radius a is tangent to $B(b)$, $B'(b)$, $C(c)$, $C'(c)$ because triangles $A''BC$ and OBC are mirror images of each other. When $b < c$, $A''(a)$ is closer to $A(a)$ than $A'(a)$. If $A''(a)$ and $A(a)$ are tangent to each other, then $AB^2 - a^2 = OB^2 - (OA-a)^2$. Now, $AB = a+b$ and $OB = a+c$, $OA = b+c$. This simplifies into $a = \frac{b(b+c)}{2c}$. If $a < \frac{b(b+c)}{2c}$, the circles $A(a)$ and $A''(a)$ are separate. □

Figure 8 shows an arrangement $(abcacb)$ with a central circle touching 5 inner circles except (A').

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