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A Theorem on Orthology Centers

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Abstract. We prove that if two triangles are orthologic, their orthology centers have the same barycentric coordinates with respect to the two triangles. For a point P with cevian triangle $A'B'C'$, we also study the orthology centers of the triangle of circumcenters of $PB'C'$, $PC'A'$, and $PA'B'$.

1. The barycentric coordinates of orthology centers

Let $A'B'C'$ be the cevian triangle of P with respect to a given triangle ABC. Denote by O_a, O_b, O_c the circumcenters of triangles $PB'C'$, $PC'A'$, $PA'B'$ respectively. Since O_bO_c, O_cO_a, and O_aO_b are perpendicular to AP, BP, CP, the triangles $O_aO_bO_c$ and ABC are orthologic at P. It follows that the perpendiculars from O_a, O_b, O_c to the sidelines BC, CA, AB are concurrent at a point Q. See Figure 1. We noted that the barycentric coordinates of Q with respect to triangle $O_aO_bO_c$ are the same as those of P with respect to triangle ABC. Alexey A. Zaslasky [7] pointed out that our original proof [3] generalizes to an arbitrary pair of orthologic triangles.

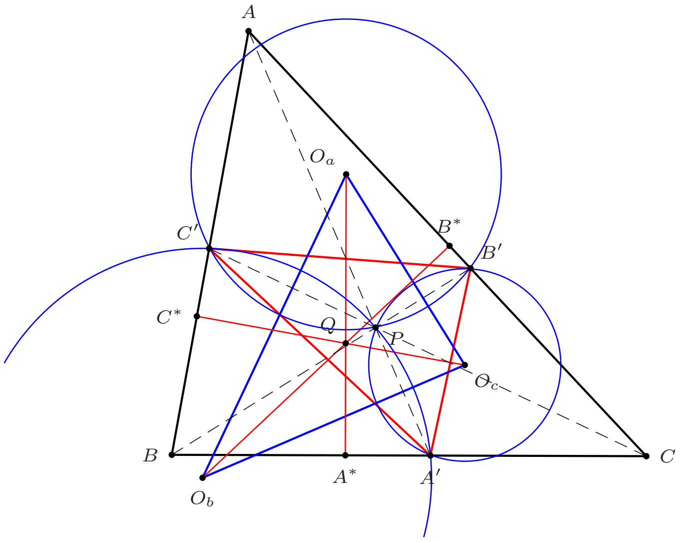

Figure 1

Theorem 1. *If triangles ABC and $A'B'C'$ are orthologic with centers P, P' then the barycentric coordinates of P with respect to ABC are equal to the barycentric coordinates of P' with respect to $A'B'C'$.*

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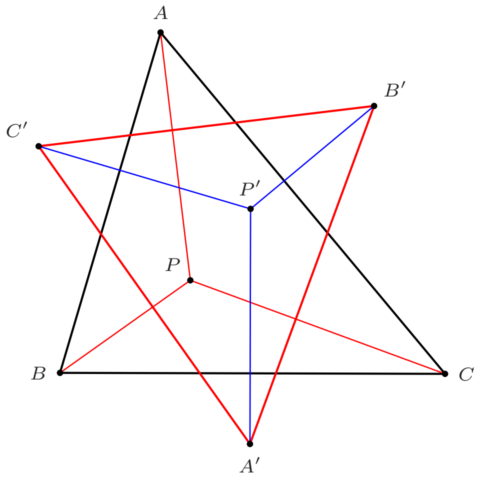

Figure 2

Proof. Since $A'P'$, $B'P'$, $C'P'$ are perpendicular to BC, CA, AB respectively, we have

$$\sin B'P'C' = \sin A, \quad \sin P'B'C' = \sin PAC, \quad \sin P'C'B' = \sin PAB.$$

Applying the law of sines to various triangles, we have

$$\begin{aligned}\frac{b}{P'B'} : \frac{c}{P'C'} =& \frac{1}{c \sin P'C'B'} : \frac{1}{b \sin P'B'C'} \\ =& \frac{1}{c \sin PAB} : \frac{1}{b \sin PAC} \\ =& \frac{1}{AP \cdot c \sin PAB} : \frac{1}{AP \cdot b \sin PAC} \\ =& \frac{1}{\text{area}(PAB)} : \frac{1}{\text{area}(PAC)} \\ =& \text{area}(PCA) : \text{area}(PAB).\end{aligned}$$

Similarly, $\frac{a}{P'A'} : \frac{b}{P'B'} = \text{area}(PBC) : \text{area}(PCA)$. It follows that the barycentric coordinates of P' with respect to triangle $A'B'C'$ are

$$\begin{aligned} & \text{area}(P'B'C') : \text{area}(P'C'A') : \text{area}(P'A'B') \\ =& (P'B')(P'C') \sin A : (P'C')(P'A') \sin B : (P'A')(P'B') \sin C \\ =& \frac{a}{P'A'} : \frac{b}{P'B'} : \frac{c}{P'C'} \\ =& \text{area}(PBC) : \text{area}(PCA) : \text{area}(PAB), \end{aligned}$$

the same as the barycentric coordinates of P with respect to triangle ABC. □

This property means that if P is the centroid of ABC then P' is also the centroid of $A'B'C'$.

2. The orthology center of $O_aO_bO_c$

We compute explicitly the coordinates (with respect to triangle ABC) of the orthology center Q of the triangle of circumcenters $O_aO_bO_c$. See Figure 3. Let $P = (x : y : z)$ and $Q = (u : v : w)$ in homogeneous barycentric coordinates. then $BC' = \frac{cx}{x+y}$, $CB' = \frac{bx}{x+z}$. In the notations of John H. Conway, the pedal A^* of O_a on BC has homogeneous barycentric coordinates $(0 : uS_C + a^2v : uS_B + a^2w)$. See, for example, [6, pp.32, 49].

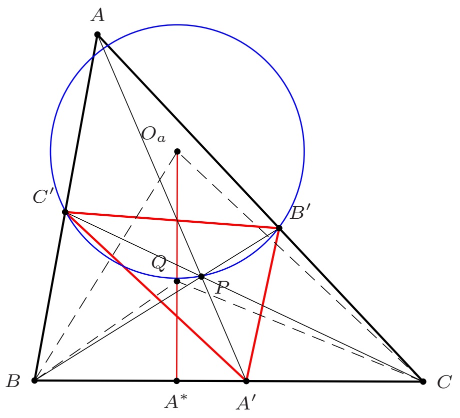

Figure 3

Note that $BA^* = \frac{uS_B+a^2w}{(u+v+w)a}$ and $A^*C = \frac{uS_C+a^2v}{(u+v+w)a}$. Also, by Stewart's theorem,

$$BB'^2 = \frac{c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)xz}{(x+z)^2},$$

$$CC'^2 = \frac{b^2x^2 + a^2y^2 + (a^2 + b^2 - c^2)xy}{(x+y)^2}.$$

Hence, if ρ is the circumradius of $PB'C'$, then

$$\begin{aligned}
&a(BA^* - A^*C)\\
=&(BA^* + A^*C)(BA^* - A^*C)\\
=&(BA^*)^2 - (A^*C)^2\\
=&(O_aB)^2 - (O_aA^*)^2 - (O_aC)^2 + (O_aA^*)^2\\
=&(O_aB)^2 - \rho^2 - (O_aC)^2 + \rho^2\\
=&BP \cdot BB' - CP \cdot CC'\\
=&\frac{c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)xz}{(x+z)(x+y+z)} - \frac{b^2x^2 + a^2y^2 + (a^2 + b^2 - c^2)xy}{(x+y)(x+y+z)}\\
=&-\frac{a^2(y-z)(x+y)(x+z) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)}
\end{aligned}$$

since the powers of B and C with respect to the circle of $PB'C'$ are $BB' \cdot BP = (O_aB)^2 - \rho^2$ and $CC' \cdot CP = (O_aC)^2 - \rho^2$ respectively. In other words,

$$\begin{aligned}&\frac{(c^2-b^2)u - a^2(v-w)}{u+v+w}\\ =&-\frac{a^2(y-z)(x+y)(x+z)+b^2x(x+y)(x+2z)-c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)},\end{aligned}$$

or

$$\begin{aligned}&(a^2(y-z)(x+y)(x+z)-b^2(x+y)(xy+yz+z^2)+c^2(x+z)(y^2+xz+yz))u\\ &-(a^2(x+y)(x+z)(x+2z)-b^2x(x+y)(x+2z)+c^2x(x+z)(x+2y))v\\ &+(a^2(x+y)(x+z)(x+2y)+b^2x(x+y)(x+2z)-c^2x(x+z)(x+2y))w=0.\end{aligned}$$

By replacing x, y, z by y, z, x and u, v, w by v,w, u, we obtain another linear relation in u, v, w. From these we have $u:v:w$ given by

$$\begin{aligned}u=&(x^2-z^2)y^2S_{BB}+(x^2-y^2)z^2S_{CC}-x(2x+y)(x+z)(y+z)S_{AB}\\ &-x(2x+z)(x+y)(y+z)S_{CA}-2(x+y)(x+z)(xy+yz+zx)S_{BC}.\end{aligned}$$

and v obtained from u by replacing x, y, z, S_A, S_B, S_C by v, w, u, S_B, S_C, S_A respectively, and w from v by the same replacements.

3. Examples

3.1. *The centroid.* For $P = G$,

$$\begin{aligned}O_a=&(5S_A(S_B+S_C)+2(S_{BB}+5S_{BC}+S_{CC})\\ &:3S_{AB}+4S_{AC}+S_{BC}-2S_{CC}\\ &:3S_{AC}+4S_{AB}+S_{BC}-2S_{BB}).\end{aligned}$$

Similarly, we write down the coordinates of O_b and O_c. The perpendiculars from O_a to BC, from O_b to CA, and from O_c to AB have equations

$$\begin{array}{rcrcrcl}(S_B-S_C)x & - & (3S_B+S_C)y & + & (S_B+3S_C)z & = & 0,\\ (S_C+3S_A)x & + & (S_C-S_A)y & - & (3S_C+S_A)z & = & 0,\\ -(3S_A+S_B)x & + & (S_A+3S_B)y & + & (S_A-S_B)z & = & 0.\end{array}$$

These three lines intersect at the nine-point center

$$X_5=(S_{CA}+S_{AB}+2S_{BC}:S_{AB}+S_{BC}+2S_{CA}:S_{BC}+S_{CA}+2S_{AB}),$$

which is the orthology center of $O_aO_bO_c$.

3.2. *The orthocenter.* If P is the orthocenter, the circumcenters O_a, O_b, O_c are simply the midpoints of the segments AP, BP, CP respectively. In this case, $Q = H$.

3.3. *The Steiner point.* If P is the Steiner point $\left(\frac{1}{S_B-S_C}:\frac{1}{S_C-S_A}:\frac{1}{S_A-S_B}\right)$, the perpendiculars from the circumcenters to the sidelines are

$$\begin{array}{rcccccl} (S_B-S_C)x & - & S_Cy & + & S_Bz & = & 0, \\ S_Cx & + & (S_C-S_A)y & - & S_Az & = & 0, \\ -S_Bx & + & S_Ay & + & (S_A-S_B)z & = & 0. \end{array}$$

These lines intersect at the deLongchamps point

$$X_{20}=(S_{CA}+S_{AB}-S_{BC}:S_{AB}+S_{BC}-S_{CA}:S_{BC}+S_{CA}-S_{AB}).$$

3.4. X_{671}. The point $P=X_{671}=\left(\frac{1}{S_B+S_C-2S_A}:\frac{1}{S_C+S_A-2S_B}:\frac{1}{S_A+S_B-2S_C}\right)$ is the antipode of the Steiner point on the Steiner circum-ellipse. It is also on the Kiepert hyperbola, with Kiepert parameter $-\operatorname{arccot}(\frac{1}{3}\cot\omega)$, where ω is the Brocard angle. In this case, the circumcenters are on the altitudes. This means that $Q=H$.

3.5. *An antipodal pair on the circumcircle.* The point X_{925} is the second intersection of the circumcircle with the line joining the deLongchamps point X_{20} to X_{74}, the isogonal conjugate of the Euler infinity point. It has coordinates

$$\left(\frac{1}{(S_B-S_C)(S^2-S_{AA})}:\frac{1}{(S_C-S_A)(S^2-S_{BB})}:\frac{1}{(S_A-S_B)(S^2-S_{CC})}\right).$$

For $P=X_{925}$, the orthology Q of $O_aO_bO_c$ is the point X_{68}, [1] which lies on the same line joining X_{20} to X_{74}.

The antipode of X_{925} is the point

$$X_{1300}=\left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_B-S_C)^2)}:\cdots:\cdots\right).$$

It is the second intersection of the circumcircle with the line joining the orthocenter to the Euler reflection point [2] $X_{110}=\left(\frac{S_B+S_C}{S_B-S_C}:\frac{S_C+S_A}{S_C-S_A}:\frac{S_A+S_B}{S_A-S_B}\right)$. For $P=X_{1300}$, the orthology center Q of $O_aO_bO_c$ has first barycentric coordinate

$$\frac{S_{AA}(S_{BB}+S_{CC})(S_A(S_B+S_C)-(S_{BB}+S_{CC}))+S_{BC}(S_B-S_C)^2(S_{AA}-2S_A(S_B+S_C)-S_{BC}))}{S_A((S_B+S_C)(S_{AA}-S_{BC})-S_A(S_B-S_C)^2)}.$$

In this case, $O_aO_bO_c$ is also perspective to ABC at

$$X_{254}=\left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_{BB}+S_{CC}))}:\cdots:\cdots\right).$$

By a theorem of Mitrea and Mitrea [5], this perspector lies on the line PQ.

[1] X_{68} is the perspector of the reflections of the orthic triangle in the nine-point center.

[2] The Euler reflection point is the intersection of the reflections of the Euler lines in the sidelines of triangle ABC.

3.6. More generally, for a generic point P on the circumcircle with coordinates $\left(\frac{S_B+S_C}{(S_A+t)(S_B-S_C)} : \cdots : \cdots\right)$, the center of orthology of $O_aO_bO_c$ is the point

$$\left(\frac{(S_B+S_C)(F(S_A,S_B,S_C)+G(S_A,S_B,S_C)t)}{S_A+t} : \cdots : \cdots\right),$$

where

$$\begin{aligned} F(S_A,S_B,S_C) =& S_{AA}(S_{BB}+S_{CC})(S_A+S_B+S_C)+S_{AABC}(S_B+S_C)\\ &-S_{BB}S_{CC}(2S_A+S_B+S_C),\\ G(S_A,S_B,S_C) =& 2(S_{AA}(S_{BB}+S_{BC}+S_{CC})-S_{BB}S_{CC}). \end{aligned}$$

Proposition 2. *If P lies on the circumcircle, the line joining P to Q always passes through the deLongchamps point X_{20}.*

Proof. The equation of the line PQ is

$$\begin{aligned} \sum_{\text{cyclic}} &(S_B-S_C)(S_A+t)(S_A^3(S_B-S_C)^2\\ &+(S_B+S_C+2t)(S_{AA}(S_{BB}-S_{BC}+S_{CC})-S_{BB}S_{CC})x=0. \end{aligned}$$

□

3.7. *Some further examples.* We conclude with a few more examples of P with relative simple coordinates for Q, the orthology center of $O_aO_bO_c$.

P	first barycentric coordinate of Q
X_7	$4a^3+a^2(b+c)-2a(b-c)^2-3(b+c)(b-c)^2$
X_8	$4a^4-5a^3(b+c)-a^2(b^2-10bc+c^2)+5a(b-c)^2(b+c)-3(b^2-c^2)^2$
X_{69}	$3a^6-4a^4(b^2+c^2)+a^2(3b^4+2b^2c^2+3c^4)-2(b^2-c^2)^2(b^2+c^2)$
X_{80}	$\frac{4a^3-3a^2(b+c)-2a(2b^2-5bc+2c^2)+3(b-c)^2(b+c)}{(b^2+c^2-a^2-bc)}$

In each of the cases $P = X_7$ and X_{80}, the triangle $O_aO_bO_c$ is also perspective to ABC at the incenter.

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