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A Grand Tour of Pedals of Conics

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Abstract. We describe the pedal curves of conics and some of their relations to origami folding axioms. There are nine basic types of pedals depending on the location of the pedal point with respect to the conic. We illustrate the different pedals in our tour.

1. Introduction

The main 'axiom' of mathematical origami allows one to create a fold-line by sliding or folding point F onto a line L so that another point S is also folded onto yet another line M. One can regard this complicated axiom as making possible the folding of the common tangents to the parabola κ with focus F and directrix L and the parabola with focus S and directrix M. Since two parabolas have at most four common tangents in the projective plane and one of them is the line at infinity there are at most three folds in the Euclidean plane which will accomplish this origami operation. In the field theory associated to origami this operation yields construction methods for solving cubic equations, [1]. Hull has shown how to do the 'impossible' trisection of an angle using this folding, by a method due to Abe, [2]. In fact the trisection of Abe is quite similar to a classical method using Maclaurin's trisectrix, [3]. The trisectrix is one of the pedals along the tour.

One can simplify this origami folding operation into smaller steps: first fold S to the point P by reflection across the tangent of the parabola κ. The locus of points P for all the tangents of κ is a curve; finally, this locus is intersected with the line M. This 'origami locus' of points P is a cubic curve since intersecting with M will generally give three possible solutions. Since reflection of S across a line is just the double of the perpendicular projection S' of S onto L, this 'origami' locus is the scale by a factor of 2 of the locus of S', also known as the pedal curve of the parabola, [3]. As a generalization we shall investigate the pedal curves of an arbitrary conic; this pedal curve is generally a quartic curve.

Pedal of a conic. The points S' of the pedal curve lie on the lines through S at the places where the tangents to the curve are perpendicular to these lines. Suppose that S is at the origin. The line through the origin perpendicular to $\alpha x + \beta y + \gamma = 0$ is the line $\beta x - \alpha y = 0$; these meet when $x = -\frac{\alpha\gamma}{\alpha^2+\beta^2}$, $y = -\frac{\beta\gamma}{\alpha^2+\beta^2}$. This suggests

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using the inversion transform (at the origin), the map given by $x \to \frac{x}{x^2+y^2},\ y \to \frac{y}{x^2+y^2}$.

A conic has the homogenous quadratic equation $F(x,y,z) = 0$ which can also be given by the matrix equation $F(x,y,z) = (x,y,z)A(x,y,z)^t = 0$ for a 3 by 3 symmetric matrix A. It is well-known that the dual curve of tangent lines to a conic is also a conic having homogeneous equation $F'(x,y,z) = 0$ obtained from the adjoint matrix A' of A. Thus the pedal curve has the (inhomogeneous) equation obtained by applying the inversion transform to $F'(x,y,-z) = 0$, evaluated at $z = 1$, [4].

The polar line of a point T is the line through the points U and V on the conic where the tangents from T meet the conic. It is important to realize the polar line of a point with respect to the conic κ having equation $F = 0$ can be expressed in terms of the matrix A. In terms of equations, if T has (projective) coordinates (u,v,w) then the dual line has the equation $(x,y,z)A(u,v,w)^t = 0$. For example, when S is placed at the origin the dual line is $(x,y,z)A(0,0,1)^t = 0$.

2. Equation of a pedal of a conic

Let S be at the origin. Suppose the (non-degenerate) conic equation is $F(x,y,z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$. Applying the inversion to the adjoint equation gives after a bit of algebra the relatively simple equation

$$G = \Delta(x^2+y^2)^2 + (x^2+y^2)((4cd-2be)x + (4ae-2bd)y) + G_2 = 0$$

where $\Delta = 4ac - b^2$ is the discriminant of the conic; $\Delta = 0$ iff the conic is a parabola. In the case of a parabola, the pedal curve has a cubic equation. The origin is a singular point having as singular tangent lines the linear factors of the degree two term $G_2 = (4cf - e^2)x^2 + (2ed - 4bf)xy + (4af - d^2)y^2 = 0$.

3. Variety of pedals

Fix a (non-empty) real conic κ in the plane and a point S. There are two points U and V on the conic with tangents τ_U and τ_V meeting at S; the corresponding pedal point for each of these tangents is S. Thus S is a double point. The type of singularity or double point at S is either a node, cusp or acnode depending on whether or not the two tangents are real and distinct, real and equal or complex conjugates.

The perpendicular lines at S to τ_U and τ_V are the singular tangents. To see this notice that the dual line to $S = (0,0)$ is $(x,y,z)A(0,0,1)^t = 0$ or equivalently $dx + ey + 2f = 0$. This line meets the conic at the points U, V which are on the tangents from S. Determining the perpendiculars through the origin S to these tangents, and multiplying the two linear factors yields after a tedious calculation precisely the second degree terms G_2 of G.

The variety of pedals depending on the type of conic and the type of singularity, are displayed in Figures 1-9, along with their associated conics, the singular point S, the singular tangents, dual line and its intersections with the conic (whenever possible).

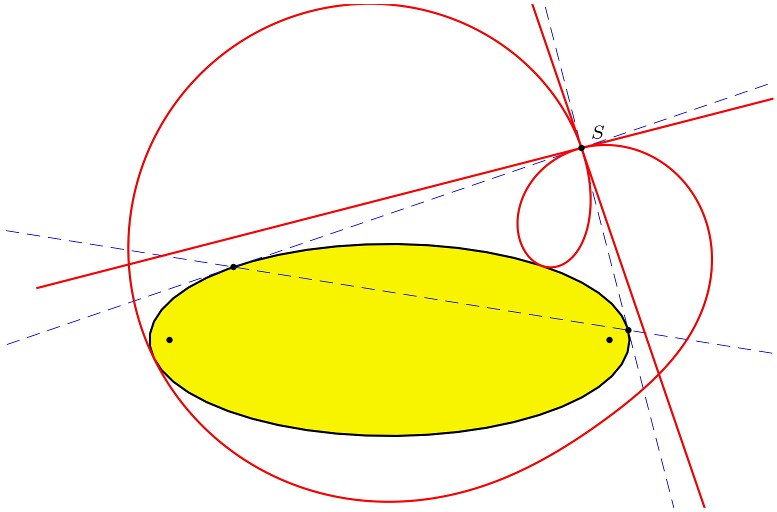

Figure 1. Elliptic node

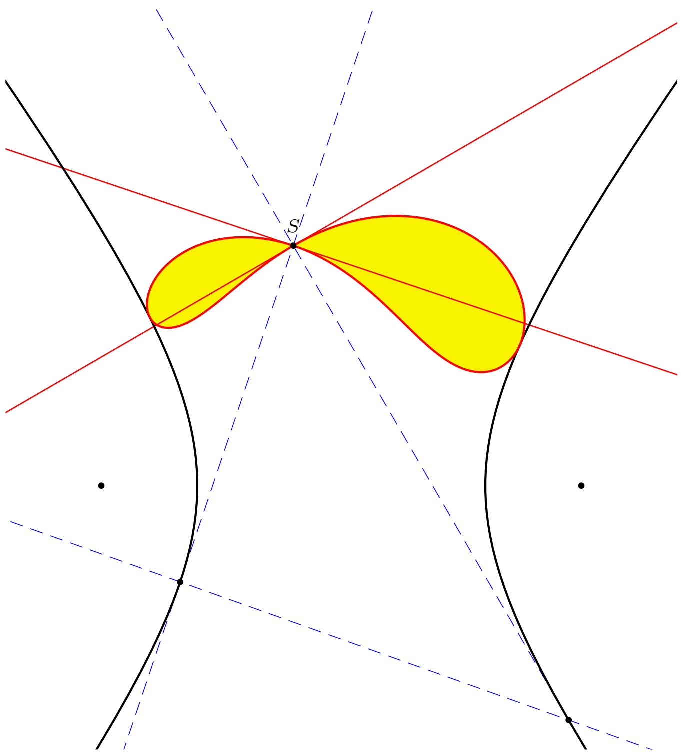

Figure 2. Hyperbolic node

Proposition 1. *The pedal of the real conic κ has a node, cusp or acnode depending on whether S is outside, on, or inside κ.*

Proof. By the calculation of the second degree terms of G, the singular tangents at the point S of the pedal are the perpendiculars to the two tangents from S to the conic κ. Thus the type of node depends on the position of S with respect to the conic since that determines how G_2 factors over the reals. $\square$

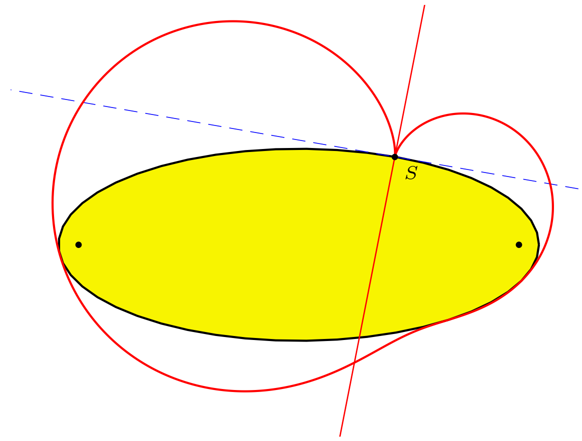

Figure 3. Elliptic cusp

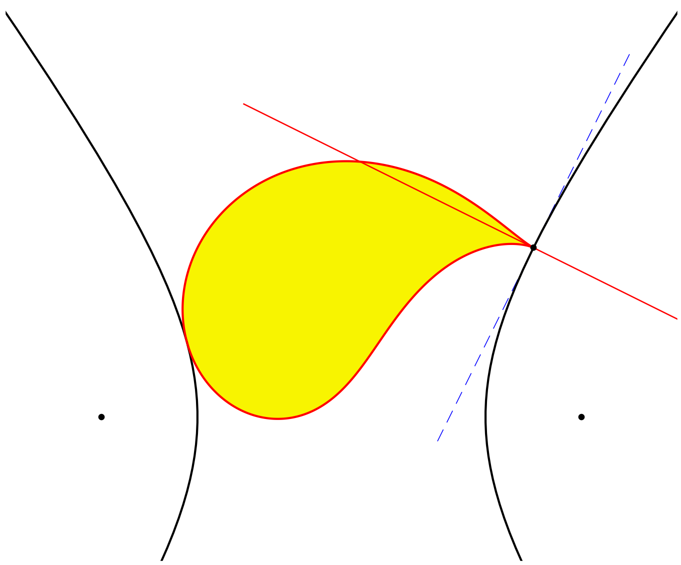

Figure 4. Hyperbolic cusp

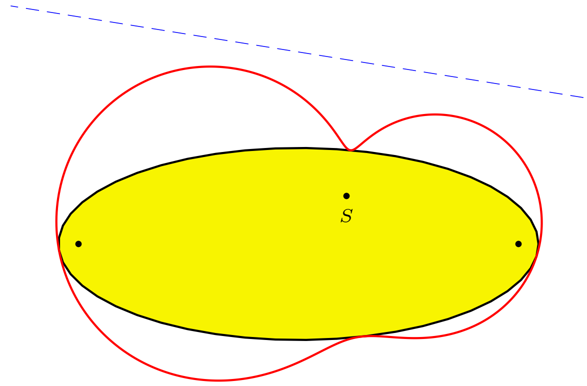

Figure 5. Elliptic acnode

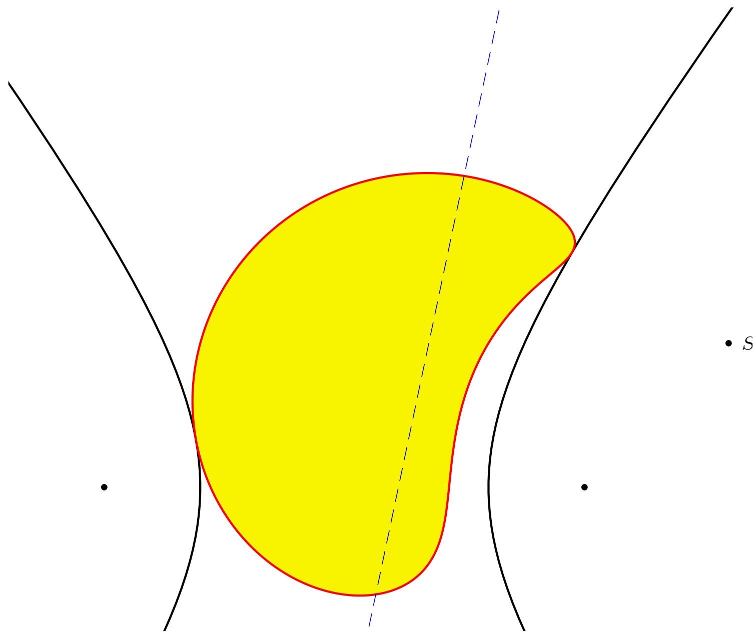

Figure 6. Hyperbolic acnode

4. Bicircular quartics

A quartic curve having circular double points is called bicircular.

Proposition 2. *A real quartic curve has the equation* $G = A(x^2+y^2)^2 + (x^2+y^2)(Bx+Cy) + Dx^2 + Exy + Fy^2 = 0$ *for* $A \neq 0$ *iff it is bicircular with double point at the origin. Thus the pedal of an ellipse or hyperbola is a bicircular quartic with a double point at* S.

Proof. A quartic has a double point at the origin iff there are no terms of degree less than 2 in the (inhomogeneous) equation $G = 0$. There are double points at the circular points iff $G(x,y,z)$ vanishes to second order when evaluated at the circular points; hence iff the gradient of G is zero at the circular points. Since $\frac{\partial G}{\partial z} = 2zG_2 + G_3$; this vanishes at the circular points iff G_3 is divisible by x^2+y^2. Also G vanishes at the circular points iff G_4 is divisible by x^2+y^2. Thus the homogeneous equation for the quartic is $G = (x^2+y^2)(ux^2+vxy+wy^2) + z(x^2+y^2)(Bx+Cy) + z^2G_2 = 0$. Finally $\frac{\partial G}{\partial x}$ or equivalently $\frac{\partial G}{\partial y}$ will also vanish at the circular points iff $ux^2+vxy+wy^2$ is divisible by x^2+y^2. Hence a bircular quartic with a double point at the origin has the equation as specified in the proposition and conversely.

The conclusion for the pedal follows immediately from the equation given in Section 2. □

We now show that any real bicircular quartic having a third double point can be realized as the pedal of a conic.

Proposition 3. *A bicircular quartic is the pedal of an ellipse or hyperbola.*

Proof. Using the equation for the pedal of a conic as in Section 2 we consider the system of equations $A = 4ac-b^2$, $B = 4cd-2be$, $C = 4ae-2bdy$, $D = 4cf-e^2$, $E = 2ed-4bf$, $F = 4af-d^2$. One can easily see that this is equivalent to a (symmetric) matrix equation $Y = X'$ where X' is the adjoint of X; we want to solve for X given Y. In our case here, Y involves the variables $A, B, \ldots$ and X involves $a, b, \ldots$ Certainly $det(Y) = det(X)^2$. Then we can solve using adjoints, $X = Y'$ iff the quadratic form $Q = Ax^2+Bxy+Cy^2+Dxz+Eyz+Fz^2$ has positive determinant. However changing G to $-G$ changes the sign of this determinant so we can represent all these quartics by pedals. □

The type of singularity of a bicircular quartic with double point at S is determined from Proposition 1 and the previous Proposition. The type of singularity of the circular double points is determined by the low order terms of G when expanded at the circular points; since the circular point is complex it is nodal in general; a circular point is cuspidal when $BC = 8AE$ and $C^2 - B^2 = 16A(D-F)$ and then in fact both circular points are cusps.

5. Pedal of parabolas

In the case that the conic is a parabola ($\Delta = 0$) the pedal equation simplifies to a cubic equation. This pedal cubic is singular and circular.

Proposition 4. *A singular circular cubic with singularity at the origin has an equation $G = (x^2+y^2)(Bx+Cy) + Dx^2 + Exy + Fy^2 = 0$ and conversely. This is the pedal of a parabola.*

Proof. The cubic is singular at the origin iff there are no terms of degree less than two; the curve is circular iff the cubic terms vanish at the circular points iff x^2+y^2 is a factor of the cubic terms.

The pedal of a parabola having $\Delta = 4ac - b^2 = 0$, means the cubic equation is $G = (x^2+y^2)((4cd-2be)x + (4ae-2bd)y) + (4cf-e^2)x^2 + (2ed-4bf)xy +$

$(4af - d^2)y^2 = 0$. Solving the system of equations as in Proposition 3 we have a simpler system since $A = 0$ but similar methods give the desired result. □

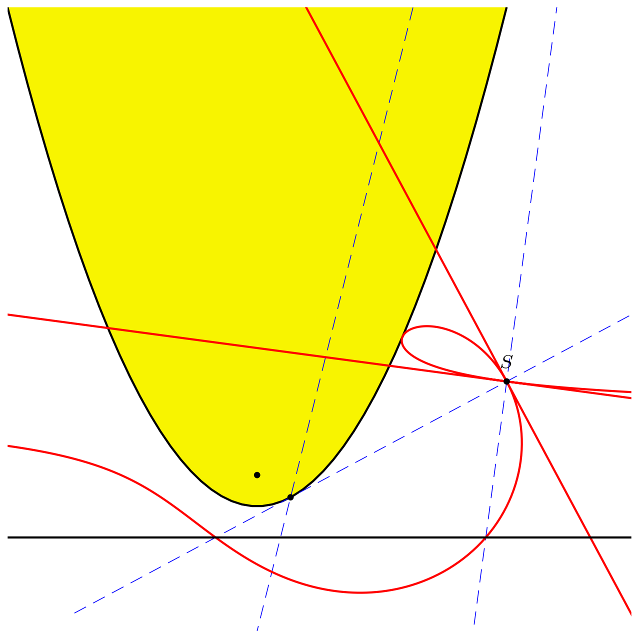

Figure 7. Parabolic node

6. Tangency of pedal and conic at their intersections

The pedal of a conic κ meets that conic at the places T iff the normal line to κ at that point passes through S. Thus the intersection occurs iff the line ST is a normal to the curve.

It follows from the fact that the conic and its pedal have a resultant which is a square (a horrendous calculation) that the pedal is tangent at all of its intersections with the conic. From Bezout's theorem, the conic and pedal have eight intersections (counted with multiplicity) and since each is a tangency there are at most four actual incidences just as expected from the figures.

Alternatively we can use elementary properties of a arbitrary curve $C(t)$ with unit speed parameterizations having tangent τ and normal η to see that when S is at the origin, the pedal $P(t)$ has a parametrization $P(t) = C(t) \cdot \eta(t)\eta(t)$ and tangent $P'(t) = -k(t)(C(t){\cdot}\tau(t)\eta(t){+}C(t){\cdot}\eta(t)\tau(t))$ where $k(t)$ is the curvature. Thus the tangent to P is parallel to τ iff $C(t) \cdot \tau(t) = 0$ iff $C(t)$ is parallel to the normal $\eta(t)$ iff the normal passes through S.

7. Linear families of pedals

Because of the importance of a parabola in the origami axioms, we illustrate in Figure 10 a family of origami curves. Recall that the origami curve is the pedal of

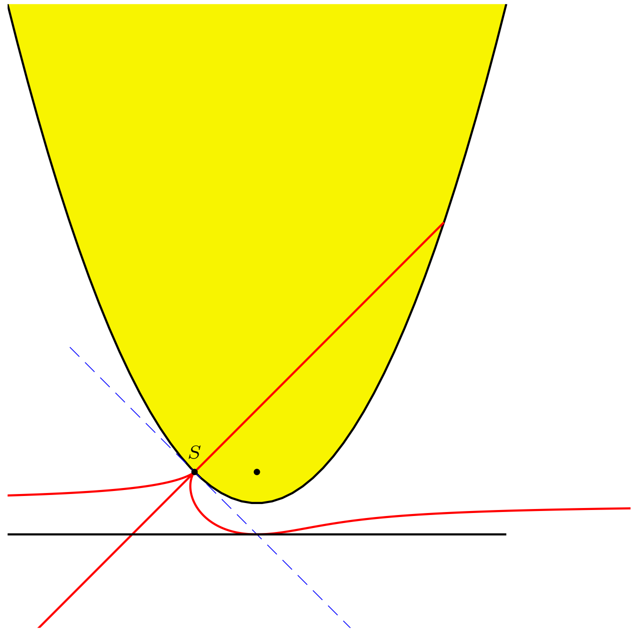

Figure 8. Parabolic cusp

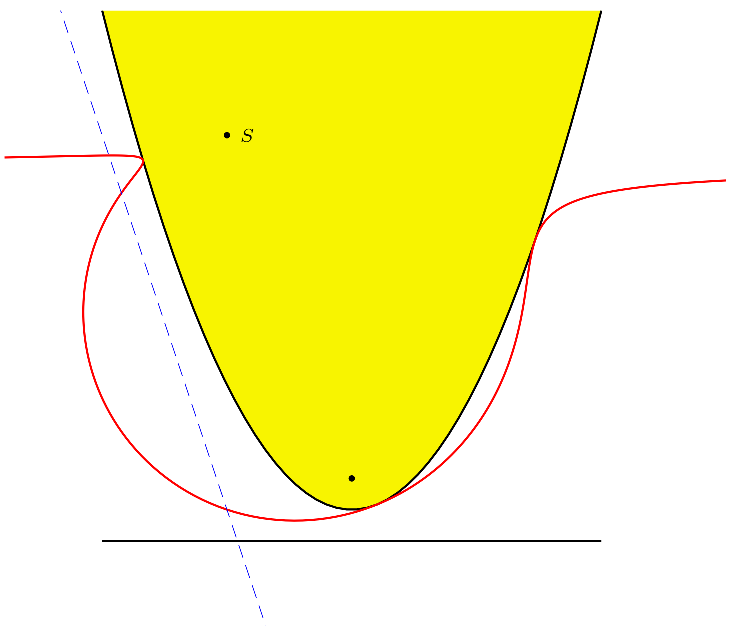

Figure 9. Parabolic acnode

a parabola scaled by 2 from the singular point S. The origami curves determined by a fixed parabola and S varying on a line parallel to the directrix are all tangent to a fixed circle of radius equal to the distance from S to the directrix. In case S varies on the directrix, then all the curves pass through the focus F.

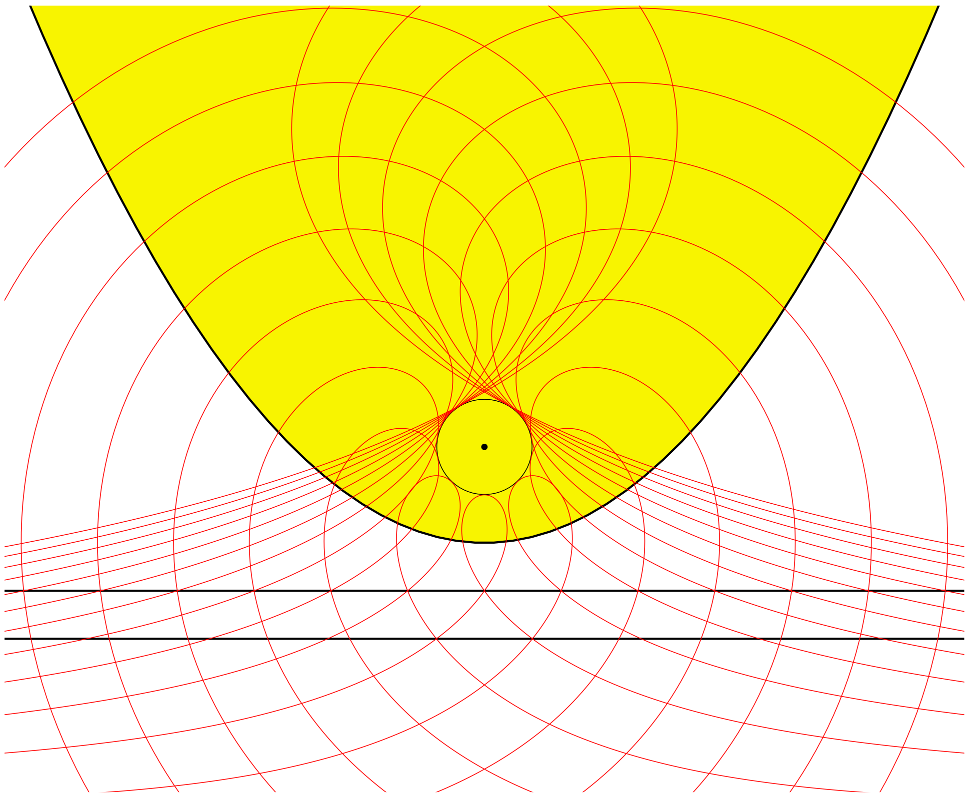

Figure 10. One parameter family of origami curves

References

[1] R. C. Alperin, A mathematical theory of origami constructions and numbers, *New York J. Math.*, 6, 119-133, 2000. *http://nyjm.albany.edu*

[2] T. Hull, A note on 'impossible' paper folding, *Amer. Math. Monthly*, 103 (199) 240–241.

[3] E. H. Lockwood, *A Book of Curves*, Cambridge University, 1963.

[4] P. Samuel, *Projective Geometry*, Springer-Verlag, 1988

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