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Generalized Mandart Conics

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Abstract. We consider interesting conics associated with the configuration of three points on the perpendiculars from a point P to the sidelines of a given triangle ABC, all equidistant from P. This generalizes the work of H. Mandart in 1894.

1. Mandart triangles

Let ABC be a given triangle and $A'B'C'$ its medial triangle. Denote by Δ, R, r the area, the circumradius, the inradius of ABC. For any $t \in \mathbb{R} \cup \{\infty\}$, consider the points P_a, P_b, P_c on the perpendicular bisectors of BC, CA, AB such that the signed distances verify $A'P_a = B'P_b = C'P_c = t$ with the following convention: for $t > 0$, P_a lies in the half-plane bounded by BC which does not contain A. We call $\mathbf{T}_t = P_aP_bP_c$ the *t-Mandart triangle* with respect to ABC. H. Mandart has studied in detail these triangles and associated conics ([5, 6]). We begin a modernized review with supplementary results, and identify the triangle centers in the notations of [4]. In the second part of this paper, we generalize the Mandart triangles and conics.

The vertices of the Mandart triangle $\mathbf{T}_t$, in homogeneous barycentric coordinates, are

$$\begin{aligned} P_a =& -ta^2 : a\Delta + tS_C : a\Delta + tS_B, \\ P_b =& b\Delta + tS_C : -tb^2 : b\Delta + tS_A, \\ P_c =& c\Delta + tS_B : c\Delta + tS_A : -tc^2, \end{aligned}$$

where

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

Proposition 1 ([6, §2]). *The points P_a, P_b, P_c are collinear if and only if $t^2 + Rt + \frac{1}{2}Rr = 0$*, i.e.,

$$t = \frac{R \pm \sqrt{R^2 - 2Rr}}{2} = \frac{R \pm OI}{2}.$$

The two lines containing those collinear points are the parallels at X_{10} (Spieker center) to the asymptotes of the Feuerbach hyperbola.

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In other words, there are exactly two sets of collinear points on the perpendicular bisectors of ABC situated at the same (signed) distance from the sidelines of ABC. See Figure 1.

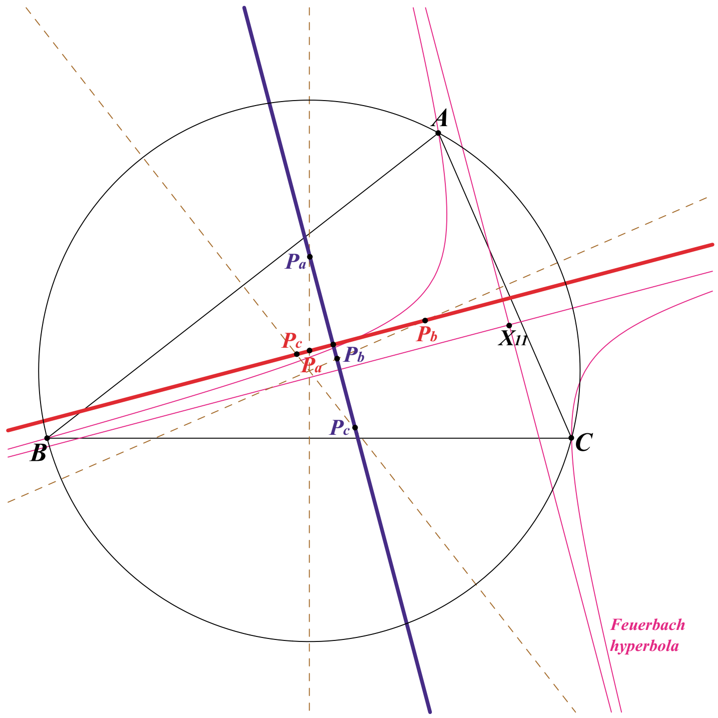

Figure 1. Collinear P_a, P_b, P_c

Proposition 2. *The triangles ABC and $P_aP_bP_c$ are perspective if and only if*
(1) $t = 0$: $P_aP_bP_c$ is the medial triangle, or
(2) $t = -r$: P_a, P_b, P_c are the projections of the incenter $I = X_1$ on the perpendicular bisectors.

In the latter case, P_a, P_b, P_c obviously lie on the circle with diameter IO. The two triangles are indirectly similar and their perspector is X_8 (Nagel point).

Remark. For any t, the triangle $Q_aQ_bQ_c$ bounded by the parallels at P_a, P_b, P_c to the sidelines BC, CA, AB is homothetic at I (incenter) to ABC.

Proposition 3. *The Mandart triangle $\mathbf{T}_t$ and the medial triangle $A'B'C'$ have the same area if and only if either :*
(1) $t = 0$: $\mathbf{T}_t$ is the medial triangle,
(2) $t = -R$,
(3) t is solution of: $t^2 + Rt + Rr = 0$.

This equation has two distinct (real) solutions when $R > 4r$, hence there are three Mandart triangles, distinct of $A'B'C'$, having the same area as $A'B'C'$. See Figure 2. In the very particular situation $R = 4r$, the equation gives the unique

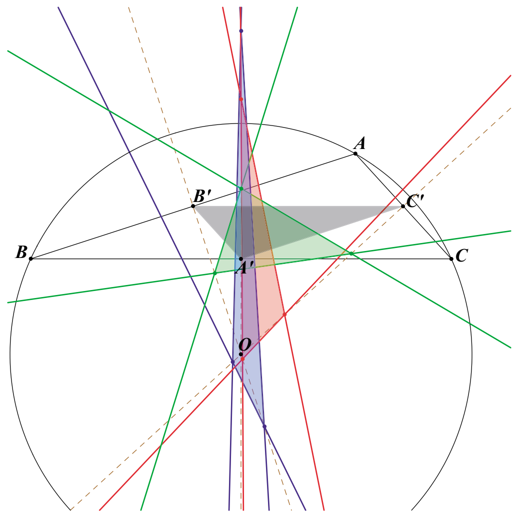

Figure 2. Three equal area triangles when $R > 4r$

solution $t = -2r = -\frac{R}{2}$ and we find only two such triangles. See Figure 3.

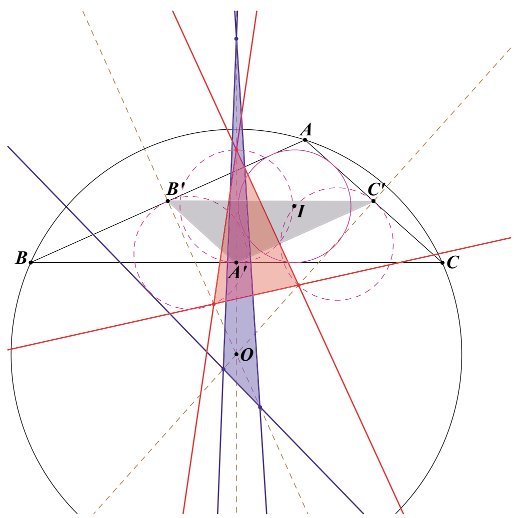

Figure 3. Only two equal area triangles when $R = 4r$

Proposition 4 ([5, §1]). *As t varies, the line P_bP_c envelopes a parabola $\mathcal{P}_a$.*

The parabola $\mathcal{P}_a$ is tangent to the perpendicular bisectors of AB and AC, to the line $B'C'$ and to the two lines met in proposition 1 above. Its focus F_a is the projection of O on the bisector AI. Its directrix ℓ_a is the bisector $A'X_{10}$ of the medial triangle. See Figure 4.

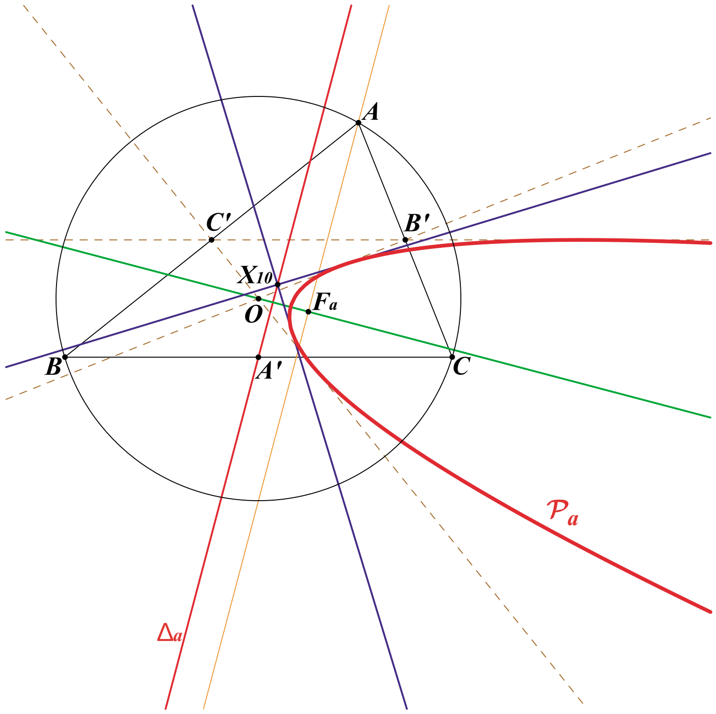

Figure 4. The parabola $\mathcal{P}_a$

Similarly, the lines P_cP_a and P_aP_b envelope parabolas $\mathcal{P}_b$ and $\mathcal{P}_c$ respectively. From this, we note the following.
(i) The foci of $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ lie on the circle with diameter OI.
(ii) The directrices concur at X_{10}.
(iii) The axes concur at O.
(iv) The contacts of the lines P_bP_c, P_cP_a, P_aP_b with $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively are collinear. See Figure 5.

These three parabolas are generally not in the same pencil of conics since their jacobian is the union of the perpendicular at O to the line IX_{10} and the circle centered at X_{10} having the same radius as the Fuhrmann circle: the polar lines of any point on this circle in the parabolas concur on the line and conversely.

2. Mandart conics

Proposition 5 ([6, §7])**.** *The Mandart triangle* $\mathbf{T}_t$ *and the medial triangle are perspective at* O*. As* t *varies, the perspectrix envelopes the parabola* $\mathcal{P}_M$ *with focus* X_{124} *and directrix* X_3X_{10}.

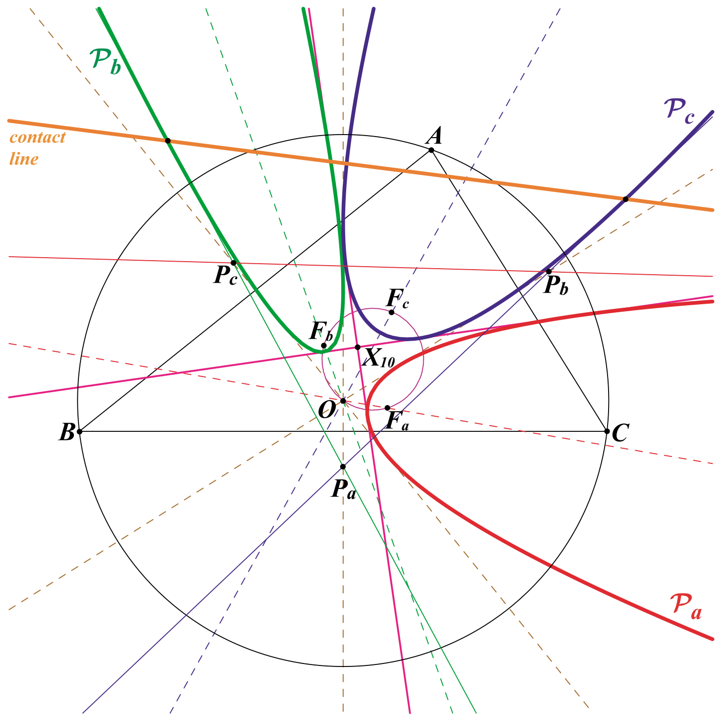

Figure 5. The three parabolas $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$

We call $\mathcal{P}_M$ the *Mandart parabola*. It has equation

$$\sum_{\text{cyclic}} \frac{x^2}{(b-c)(b+c-a)} = 0.$$

Triangle ABC is clearly self-polar with respect to $\mathcal{P}_M$. The directrix is the line X_3X_{10} and the focus is X_{124}. $\mathcal{P}_M$ is inscribed in the medial triangle with perspector

$$X_{1146} = ((b-c)^2(b+c-a)^2 : \cdots : \cdots),$$

the center of the circum-hyperbola passing through G and X_8 with respect to this triangle. The contacts of $\mathcal{P}_M$ with the sidelines of the medial triangle lie on the perpendiculars dropped from A, B, C to the directrix X_3X_{10}. $\mathcal{P}_M$ is the complement of the inscribed parabola with focus X_{109} and directrix the line IH. See Figure 6.

Proposition 6 ([5, 2, p.551])**.** *The Mandart triangle* $\mathbf{T}_t$ *and* ABC *are orthologic. The perpendiculars from* A, B, C *to the corresponding sidelines of* $P_aP_bP_c$ *are concurrent at*

$$Q_t = \left(\frac{a}{aS_A + 4\Delta t} : \cdots : \cdots\right).$$

As t *varies, the locus of* Q_t *is the Feuerbach hyperbola.*

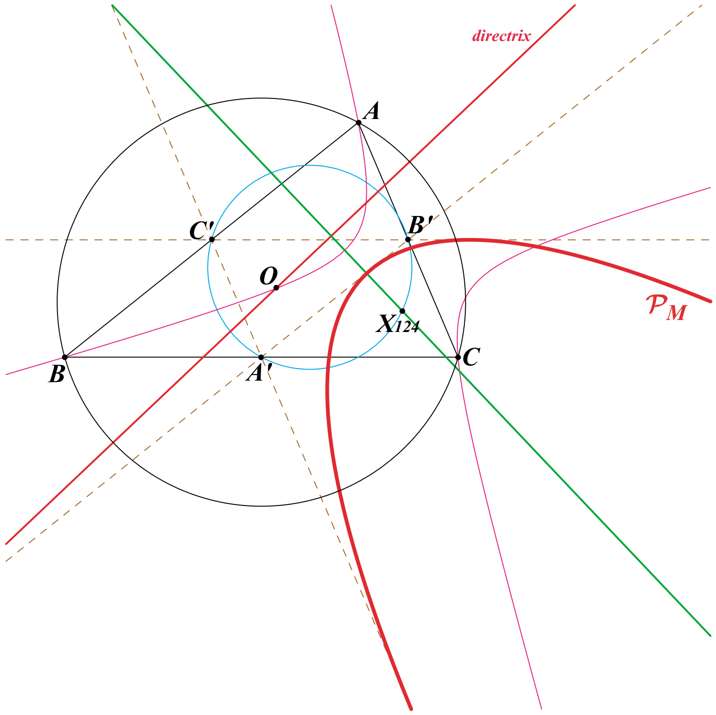

Figure 6. The Mandart parabola

Remark. The triangles $A'B'C'$ and $\mathbf{T}_t$ are also orthologic at Q'_t, the complement of Q_t.

Denote by $A_1B_1C_1$ the extouch triangle (see [3, p.158, §6.9]), *i.e.*, the cevian triangle of X_8 (Nagel point) or equivalently the pedal triangle of X_{40} (reflection of I in O). The circumcircle $\mathcal{C}_M$ of $A_1B_1C_1$ is called *Mandart circle*. $\mathcal{C}_M$ is therefore the pedal circle of X_{40} and X_{84} (isogonal conjugate of X_{40}), the cevian circumcircle of X_{189} (cyclocevian conjugate of X_8). $\mathcal{C}_M$ contains the Feuerbach point X_{11}. Its center is X_{1158}, intersection of the lines X_1X_{104} and X_8X_{40}. The second intersection with the incircle is X_{1364} and the second intersection with the nine-point circle is the complement of X_{934}. See Figure 7. The *Mandart ellipse* $\mathcal{E}_M$ (see [6, §§3,4]) is the inscribed ellipse with center X_9 (Mittenpunkt) and perspector X_8. It contains A_1, B_1, C_1, X_{11} and its axes are parallel to the asymptotes of the Feuerbach hyperbola. See Figure 7.

The equation of $\mathcal{E}_M$ is:

$$\sum_{\text{cyclic}} (c+a-b)^2(a+b-c)^2x^2 - 2(b+c-a)^2(c+a-b)(a+b-c)yz = 0$$

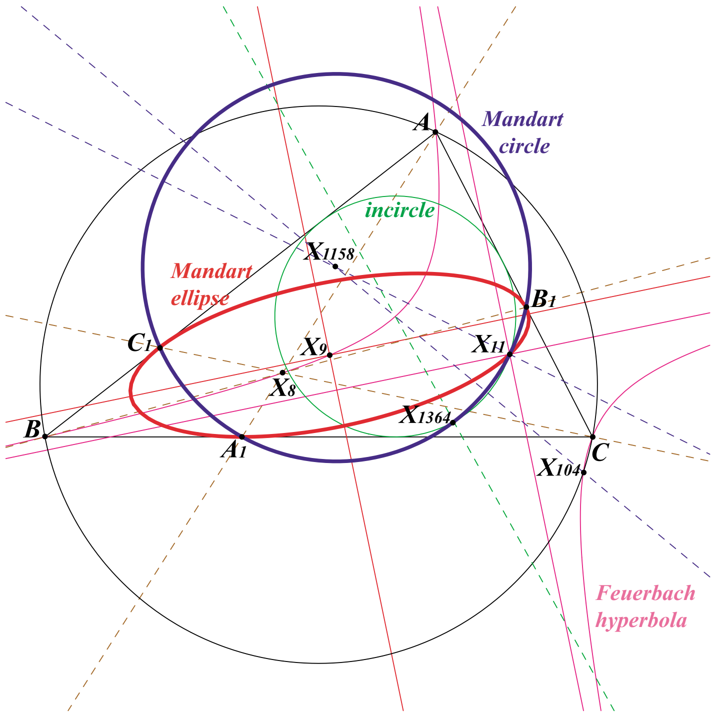

Figure 7. The Mandart circle and the Mandart ellipse

From this, we see that $\mathcal{C}_M$ is the Joachimsthal circle of X_{40} with respect to $\mathcal{E}_M$: the four normals drawn from X_{40} to $\mathcal{E}_M$ pass through A_1, B_1, C_1 and

$$F' = ((b+c-a)((b-c)^2 + a(b+c-2a))^2 : \cdots : \cdots),$$

the reflection X_{11} in X_9. [1]

The radical axis of $\mathcal{C}_M$ and the nine-point circle is the tangent at X_{11} to $\mathcal{E}_M$ and also the polar line of G in $\mathcal{P}_M$. The projection of X_9 on this tangent is the point X_{1364} we met above. Hence, $\mathcal{C}_M$, the nine-point circle and the circle with diameter X_9X_{11} belong to the same pencil of (coaxal) circles ([6, §§8,9]).

The radical axis of $\mathcal{C}_M$ and the incircle is the polar line of X_{10} in $\mathcal{P}_M$.

Proposition 7. [6, §§1,2] *The Mandart triangle $\mathbf{T}_t$ and the extouch triangle are orthologic. The perpendiculars drawn from A_1, B_1, C_1 to the corresponding sidelines of $\mathbf{T}_t = P_aP_bP_c$ are concurrent at S. As t varies, the locus of S is the rectangular hyperbola $\mathcal{H}_M$ passing through the traces of X_8 and $X_{190} = \left(\frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}\right)$*

We call $\mathcal{H}_M$ the *Mandart hyperbola*. It has equation

$$\sum_{\text{cyclic}} (b-c)\left[(c+a-b)(a+b-c)x^2 + (b+c-a)^2 yz\right] = 0$$

[1] This point is not in the current edition of [4].

and contains the triangle centers X_8, X_9, X_{40}, X_{72}, X_{144}, X_{1145}, F', and F'' antipode of X_{11} on $\mathcal{C}_M$. Its asymptotes are parallel to those of the Feuerbach hyperbola. $\mathcal{H}_M$ is the Apollonian hyperbola of X_{40} with respect to $\mathcal{E}_M$. See Figure 8.

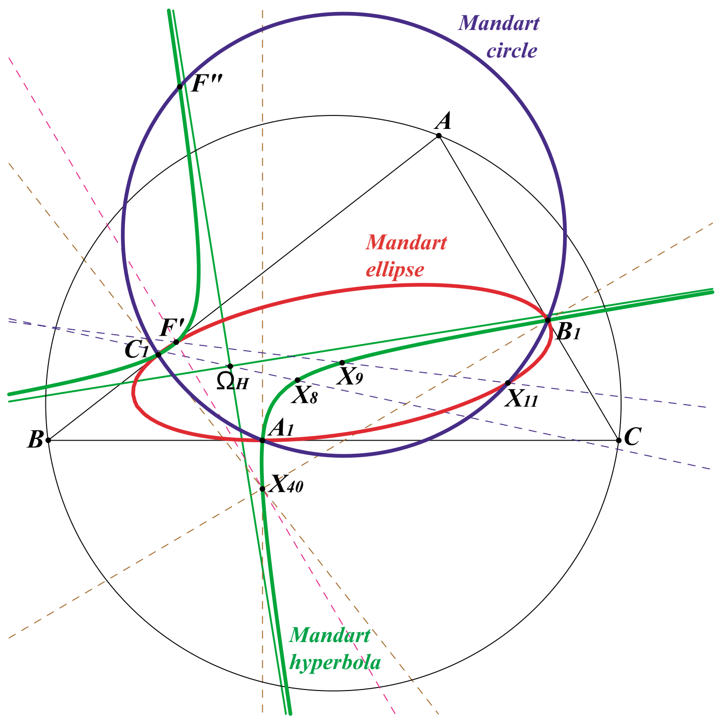

Figure 8. The Mandart hyperbola

3. Locus of some triangle centers in the Mandart triangles

We now examine the locus of some triangle centers of $\mathbf{T}_t = P_aP_bP_c$ when t varies. We shall consider the centroid, circumcenter, orthocenter, and Lemoine point.

Proposition 8. *The locus of the centroid of $\mathbf{T}_t$ is the parallel at G to the line OI.*

Proposition 9. *The locus of the circumcenter of $\mathbf{T}_t$ is the rectangular hyperbola passing through X_1, X_5, X_{10}, X_{21} (Schiffler point) and X_{1385}.* [2]

The equation of the hyperbola is

$$\sum_{\text{cyclic}} (b-c)\left[bc(b+c)x^2 + a(b^2+c^2-a^2+3bc)yz\right] = 0.$$

[2] X_{1385} is the midpoint of OI.

It has center X_{1125} (midpoint of IX_{10}) and asymptotes parallel to those of the Feuerbach hyperbola.

The locus of the orthocenter of $\mathbf{T}_t$ is a nodal cubic with node X_{10} passing through O, X_{1385}, meeting the line at infinity at X_{517} and the infinite points of the Feuerbach hyperbola. The line through the orthocenters of the t-Mandart triangle and the $(-t)$-Mandart triangle passes through a fixed point.

The locus of the Lemoine point of $\mathbf{T}_t$ is another nodal cubic with node X_{10}.

4. Generalized Mandart conics

Most of the results above can be generalized when X_8 is replaced by any point M on the Lucas cubic, the isotomic cubic with pivot X_{69}. The cevian triangle of such a point M is the pedal triangle of a point N on the Darboux cubic, the isogonal cubic with pivot the de Longchamps point X_{20}. [3]

For example, with $M = X_8$, we find $N = X_{40}$ and $M' = X_1 = I$.

Denote by $M_aM_bM_c$ the cevian triangle of M (on the Lucas cubic) and the pedal triangle of N (on the Darboux cubic). N^* is the isogonal conjugate of N also on the Darboux cubic. We now consider

– γ_M, inscribed conic in ABC with perspector M and center ω_M, which is the complement of the isotomic conjugate of M. It lies on the Thomson cubic and on the line KM' ($K = X_6$ is the Lemoine point),

– Γ_M, circumcircle of $M_aM_bM_c$ with center Ω_M, midpoint of NN^*. Γ_M is obviously the pedal circle of N and N^* and also the cevian circle of M°, cyclocevian conjugate of M (see [3, p.226, §8.12]). M° is a point on the Lucas cubic since this cubic is invariant under cyclocevian conjugation.

Since γ_M and Γ_M have already three points in common, they must have a fourth (always real) common point Z. Finally, denote by Z' the reflection of Z in ω_M. See Figure 9.

Table 1 gives examples for several known centers M on the Lucas cubic.[4] Those marked with $*$ are indicated in Table 2; those marked with ? are too complicated to give here.

Table 1

M	X_8	X_2	X_4	X_7	X_{20}	X_{69}	X_{189}	X_{253}	X_{329}	X_{1032}	X_{1034}
N	X_{40}	X_3	X_4	X_1	X_{1498}	X_{20}	X_{84}	X_{64}	X_{1490}	$*$	$*$
M'	X_1	X_2	X_3	X_9	X_4	X_6	X_{223}	X_{1249}	X_{57}	$*$	$*$
N^*	X_{84}	X_4	X_3	X_1	$*$	X_{64}	X_{40}	X_{20}	$*$	X_{1498}	X_{1490}
M°	X_{189}	X_4	X_2	X_7	X_{1032}	X_{253}	X_8	X_{69}	X_{1034}	X_{20}	X_{329}
ω_M	X_9	X_2	X_6	X_1	X_{1249}	X_3	X_{57}	X_4	X_{223}	X_{1073}	X_{282}
Ω_M	X_{1158}	X_5	X_5	X_1	?	?	X_{1158}	?	?	?	?
Z	X_{11}	X_{115}	X_{125}	X_{11}	X_{122}	X_{125}	$*$	X_{122}	$*$	?	$*$
Z'	$*$	$*$	$*$	X_{1317}	$*$	$*$	$*$	$*$	$*$	?	$*$

[3]It is also known that the complement of M is a point M' on the the Thomson cubic, the isogonal cubic with pivot $G = X_2$, the centroid.

[4]Two isotomic conjugates on the Lucas cubic are associated to the same point Z on the nine-point circle.

Table 2

Triangle center	First barycentric coordinate
$Z'(X_8)$	$(b+c-a)(2a^2-a(b+c)-(b-c)^2)^2$
$Z'(X_2)$	$(2a^2-b^2-c^2)^2$
$Z'(X_4)$	$\frac{(2a^2-b^2-c^2)^2}{S_A}$
$Z'(X_{20})$	$((3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)$ $\cdot(2a^8-a^6(b^2+c^2)-5a^4(b^2-c^2)^2+5a^2(b^2-c^2)^2(b^2+c^2)$ $-(b^2-c^2)^2(b^4+6b^2c^2+c^4))^2$
$Z'(X_{69})$	$S_A(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)^2$
$Z(X_{189})$	$(b-c)^2(b+c-a)^2(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2$
$Z'(X_{189})$	$\frac{(2a^2-a(b+c)-(b-c)^2)^2}{a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2}$
$Z'(X_{253})$	$\frac{(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)^2}{3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2}$
$Z(X_{329})$	$(b-c)^2(b+c-a)^2(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2$
$Z'(X_{329})$	$(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$ $\cdot(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b-c)^2(b+c)^3)^2$
$N^*(X_{20})$	$1/(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))$
$N^*(X_{329})$	$a/(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b+c)(b-c)^2(b^2+c^2)+(b-c)^2(b+c)^4)$
$N(X_{1032})$	$1/(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))$
$M'(X_{1032})$	$(a^2(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))/$
	$(3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)$
$N(X_{1034})$	$a/(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$
	$-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)$
$M'(X_{1034})$	$a(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)/$ $(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$
$Z(X_{1034})$	$(b-c)^2(b+c-a)(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)^2$ $\cdot(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4))$
$Z'(X_{1034})$	$(b+c-a)(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b^2-c^2)^3)^2)/(a^6-2a^5(b+c)-a^4(b+c)^2$ $+4a^3(b+c)(b^2-bc+c^2)-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)$ $+(b-c)^2(b+c)^4)$
$M'(X_{1034})$	$a(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)/$ $(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$
$Z(X_{1034})$	$(b-c)^2(b+c-a)(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)^2$ $\cdot(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4))$
$Z'(X_{1034})$	$(b+c-a)(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b^2-c^2)^3)^2)/(a^6-2a^5(b+c)-a^4(b+c)^2$ $+4a^3(b+c)(b^2-bc+c^2)-a^2(b^2-c^2)^2$ $-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)$

Proposition 10. *Z is a point on the nine-point circle and Z' is the foot of the fourth normal drawn from N to γ_M.*

Proof. The lines NM_a, NM_b, NM_c are indeed already three such normals hence Γ_M is the Joachimsthal circle of N with respect to γ_M. This yields that Γ_M must pass through the reflection in ω_M of the foot of the fourth normal. See Figure 9. □

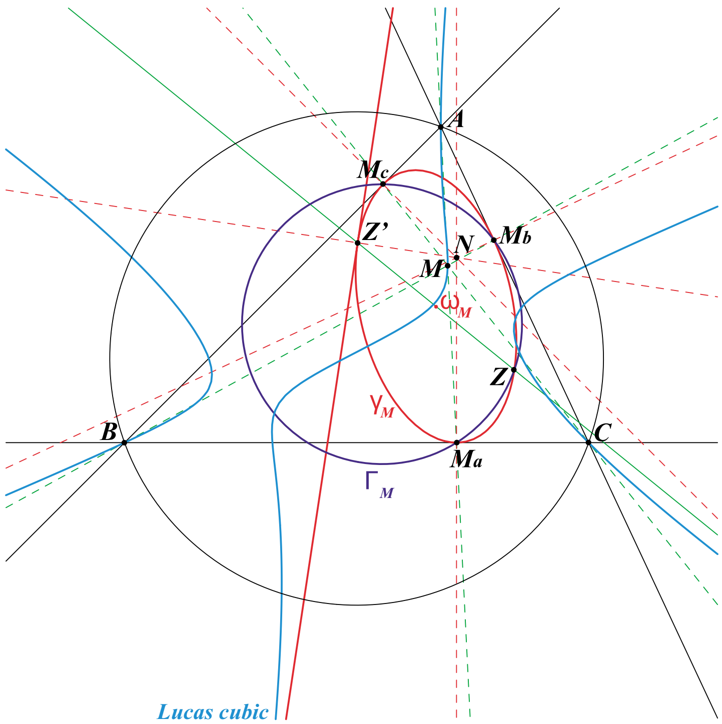

Figure 9. The generalized Mandart circle and conic

Remark. Z also lies on the cevian circumcircle of $M^{\#}$ isotomic conjugate of M and on the inscribed conic with perspector $M^{\#}$ and center M'.

Proposition 11. *The points M_a, M_b, M_c, M, N, ω_M and Z' lie on a same rectangular hyperbola whose asymptotes are parallel to the axes of γ_M.*

Proof. This hyperbola is the Apollonian hyperbola of N with respect to γ_M. □

Proposition 12. *The rectangular hyperbola passing through A, B, C, H and M is centered at Z. It also contains M', N^*, ω_M and $M^{\#}$. Its asymptotes are also parallel to the axes of γ_M.*

Remark. This hyperbola is the isogonal transform of the line ON and the isotomic transform of the line $X_{69}M$.

5. Generalized Mandart triangles

We now replace the circumcenter O by any finite point $P = (u : v : w)$ not lying on one sideline of ABC and we still call $A'B'C'$ its pedal triangle. For $t \in \mathbb{R} \cup \{\infty\}$, consider P_a, P_b, P_c defined as follows: draw three parallels to BC, CA, AB at the (signed) distance t with the conventions at the beginning of the paper. P_a, P_b, P_c are the projections of P on these parallels. See Figure 10.

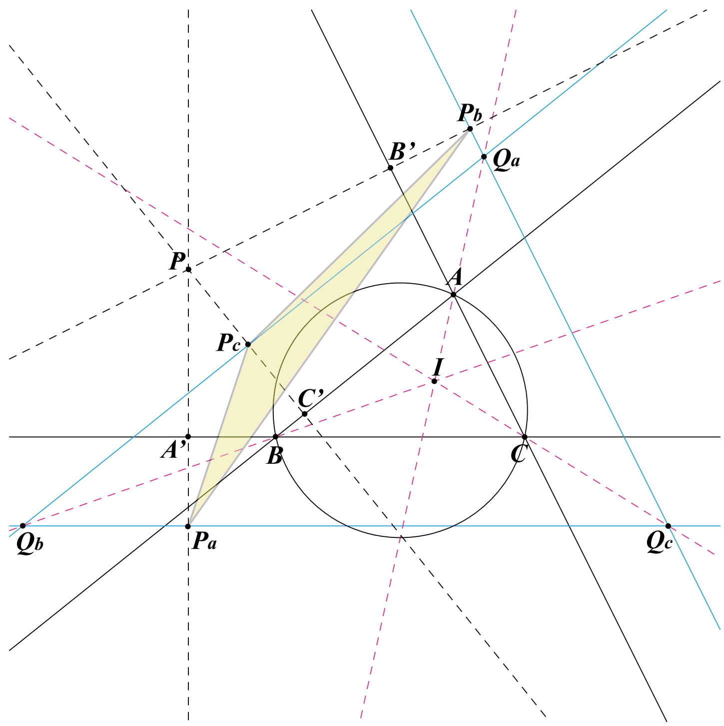

Figure 10. Generalized Mandart triangle

In homogeneous barycentric coordinates, these are the points

$$\begin{aligned}
P_a =& - a^3 t : 2\Delta \cdot \frac{S_C u + a^2 v}{u+v+w} + taS_C : 2\Delta \cdot \frac{S_B u + a^2 w}{u+v+w} + taS_B, \\
P_b =& 2\Delta \cdot \frac{S_C v + b^2 u}{u+v+w} + tbS_C : -b^3 t : 2\Delta \cdot \frac{S_A v + b^2 w}{u+v+w} + tbS_A, \\
P_c =& 2\Delta \cdot \frac{S_B w + c^2 u}{u+v+w} + tcS_B : 2\Delta \cdot \frac{S_A w + c^2 v}{u+v+w} + tcS_A : -c^3 t.
\end{aligned}$$

The triangle $\mathbf{T}_t(P) = P_a P_b P_c$ is called $t-$Mandart triangle of P.

Proposition 13. *For any P distinct from the incenter I, there are always two sets of collinear points P_a, P_b, P_c. The two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ containing the points are* *parallel to the asymptotes of the hyperbola which is the isogonal conjugate of the parallel to IP at X_{40}*[5]*. They meet at the point :*

$$(a((b+c)bcu + cS_C v + bS_B w) : \cdots : \cdots).$$

They are perpendicular if and only if P lies on OI.

Proof. P_a, P_b, P_c are collinear if and only if t is solution of the equation :

$$abc(a+b+c)t^2 + 2\Delta\, \Phi_1(u,v,w)\, t + 4\Delta^2\, \Phi_2(u,v,w) = 0 \qquad (1)$$

where

$$\Phi_1(u,v,w) = \sum_{\text{cyclic}} bc(b+c)u \quad \text{and} \quad \Phi_2(u,v,w) = \sum_{\text{cyclic}} a^2vw.$$

We notice that $\Phi_1(u,v,w) = 0$ if and only if P lies on the polar line of I in the circumcircle and $\Phi_2(u,v,w) = 0$ if and only if P lies on the circumcircle.

The discriminant of (1) is non-negative for all P and null if and only if $P = I$. In this latter case, the points P_a, P_b, P_c are "collinear" if and only if they all coincide with I.

Considering now $P \neq I$, (1) always has two (real) solutions. □

Figure 11 shows the case $P = H$ with two (non-perpendicular) lines secant at X_{65} orthocenter of the intouch triangle.

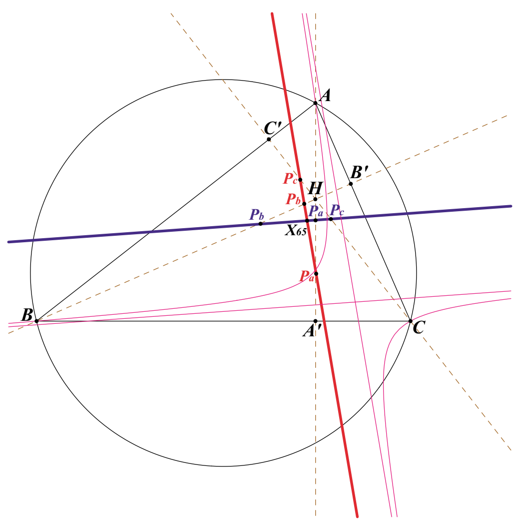

Figure 11. Collinear P_a, P_b, P_c with $P = H$

Figure 12 shows the case $P = X_{40}$ with two perpendicular lines secant at X_8 and parallel to the asymptotes of the Feuerbach hyperbola.

When P is a point on the circumcircle, equation (1) has a solution $t = 0$ and one of the two lines, say $\mathcal{L}_1$, is the Simson line of P: the triangle $A'B'C'$ degenerates

[5] X_{40} is the reflection of I in O.

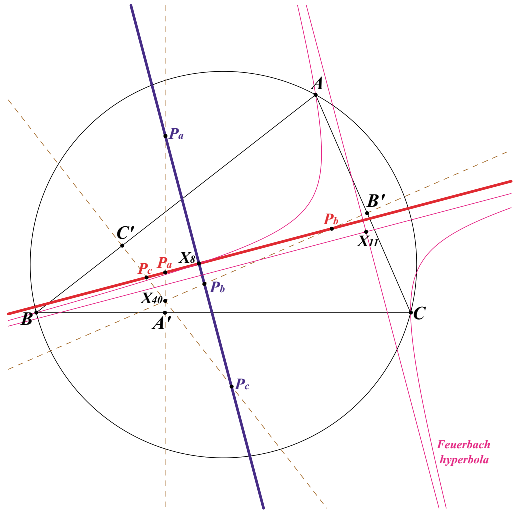

Figure 12. Collinear P_a, P_b, P_c with $P = X_{40}$

into this Simson line. $\mathcal{L}_1$ and $\mathcal{L}_2$ meet on the ellipse centered at X_{10} passing through X_{11}, the midpoints of ABC and the feet of the cevians of X_8. This ellipse is the complement of the circum-ellipse centered at I and has equation :

$$\sum_{\text{cyclic}} (a+b-c)(a-b+c)x^2 - 2a(b+c-a)yz = 0.$$

Figure 13 shows the case $P = X_{104}$ with two lines secant at X_{11}, one of them being the Simson line of X_{104}.

Following equation (1) again, we observe that, when P lies on the polar line of I in the circumcircle, we find to opposite values for t: the two corresponding points P_a are symmetric with respect to the sideline BC, P_b and P_c similarly. The most interesting situation is obtained with $P = X_{36}$ (inversive image of I in the circumcircle) since we find two perpendicular lines $\mathcal{L}_1$ and $\mathcal{L}_2$, parallel to the asymptotes of the Feuerbach hyperbola, intersecting at the midpoint of $X_{36}X_{80}$[6]. See Figure 14.

Construction of $\mathcal{L}_1$ and $\mathcal{L}_2$: the line IP[7] meets the circumcircle at S_1 and S_2. The parallels at P to OS_1 and OS_2 meet OI at T_1 and T_2. The homotheties with center I which map O to T_1 and T_2 also map the triangle ABC to the triangles $A_1B_1C_1$ and $A_2B_2C_2$. The perpendiculars PA', PB', PC' at P to the sidelines of ABC meet the corresponding sidelines of $A_1B_1C_1$ and $A_2B_2C_2$ at the requested points.

[6] X_{80} is the isogonal conjugate of X_{36}.
[7] We suppose $I \neq P$.

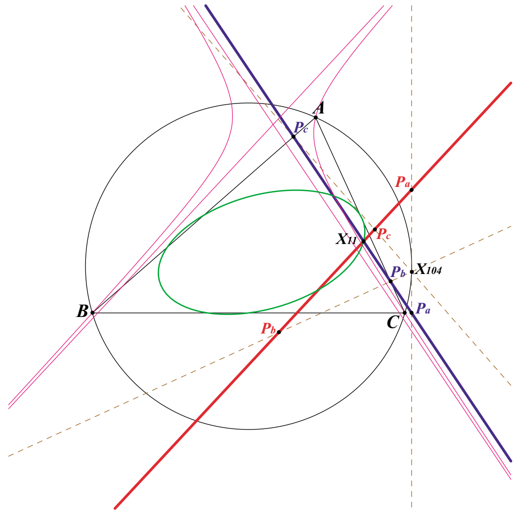

Figure 13. Collinear P_a, P_b, P_c with $P = X_{104}$

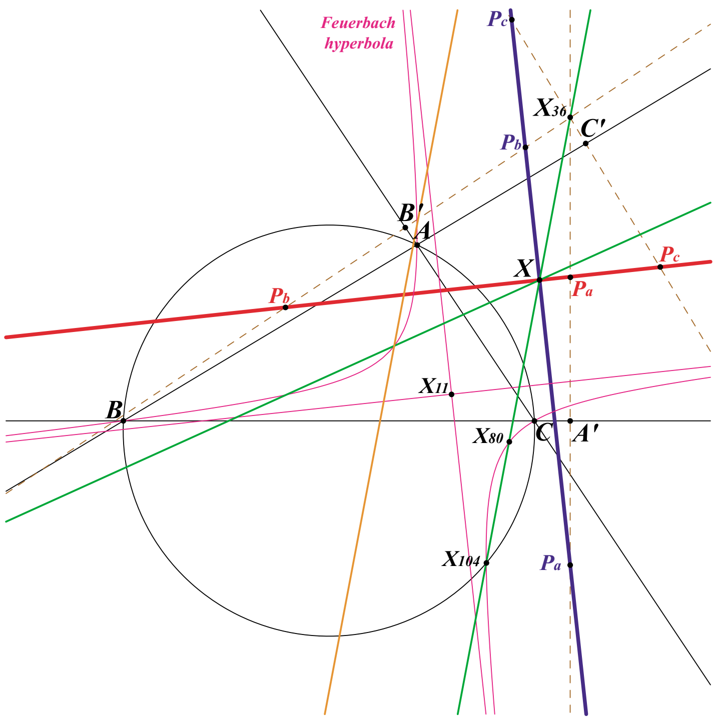

Figure 14. Collinear P_a, P_b, P_c with $P = X_{36}$

Proposition 14. *The triangles ABC and $P_aP_bP_c$ are perspective if and only if k is solution of :*

$$\Psi_2(u,v,w)\,t^2+\Psi_1(u,v,w)\,t+\Psi_0(u,v,w)=0 \tag{2}$$

where :

$$\Psi_2(u,v,w)=-\frac{1}{2}abc(a+b+c)(u+v+w)^2\sum_{\text{cyclic}}(b-c)(b+c-a)S_Au,$$

$$\Psi_1(u,v,w)=\frac{1}{2}(a+b+c)(u+v+w)\Delta\sum_{\text{cyclic}}\left(-2bc(b-c)(b+c-a)S_Au^2\right.$$
$$\left.+\,a^2(b-c)(a+b+c)(b+c-a)^2vw\right),$$

$$\Psi_0(u,v,w)=\Delta^2\sum_{\text{cyclic}}(3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)u(c^2v^2-b^2w^2).$$

Remarks. (1) $\Psi_2(u,v,w)=0$ if and only if P lies on the line IH.

(2) $\Psi_1(u,v,w)=0$ if and only if P lies on the hyperbola passing through I, H, X_{500}, X_{573}, X_{1742} [8] and having the same asymptotic directions as the isogonal transform of the line $X_{40}X_{758}$, *i.e.*, the reflection in O of the line X_1X_{21}.

(3) $\Psi_0(u,v,w)=0$ if and only if P lies on the Darboux cubic. See Figure 15.

The equation (2) is clearly realized for all t if and only if $P=I$ or $P=H$: all $t-$Mandart triangles of I and H are perspective to ABC. Furthermore, if $P=H$ the perspector is always H, and if $P=I$ the perspector lies on the Feuerbach hyperbola. In the sequel, we exclude those two points and see that there are at most two real numbers t_1 and t_2 for which t_1- and t_2-Mandart triangles of P are perspective to ABC. Let us denote by R_1 and R_2 the (not always real) corresponding perspectors.

We explain the construction of these two perspectors with the help of several lemmas.

Lemma 15. *For a given P and a corresponding Mandart triangle $\mathbf{T}_t(P)=P_aP_bP_c$, the locus of $R_a=BP_b\cap CP_c$, when t varies, is a conic γ_a.*

Proof. The correspondence on the pencils of lines with poles B and C mapping the lines BP_b and CP_c is clearly an involution. Hence, the common point of the two lines must lie on a conic. □

This conic γ_a obviously contains B, C, H, $S_a=BB'\cap CC'$ and two other points B_1 on AB, C_1 on AC defined as follows. Reflect $AB\cap PB'$ in the bisector AI to get a point B_2 on AC. The parallel to AB at B_2 meets PC' at B_3. B_1 is the intersection of AB and CB_3. The point C_1 on AC is constructed similarly. See Figure 16.

Lemma 16. *The three conics γ_a, γ_b, γ_c have three points in common: H and the (not always real) sought perspectors R_1 and R_2. Their jacobian must degenerate*

[8] $X_{500}=X_1X_{30}\cap X_3X_6$, $X_{573}=X_4X_9\cap X_3X_6$ and $X_{1742}=X_1X_7\cap X_3X_{238}$.

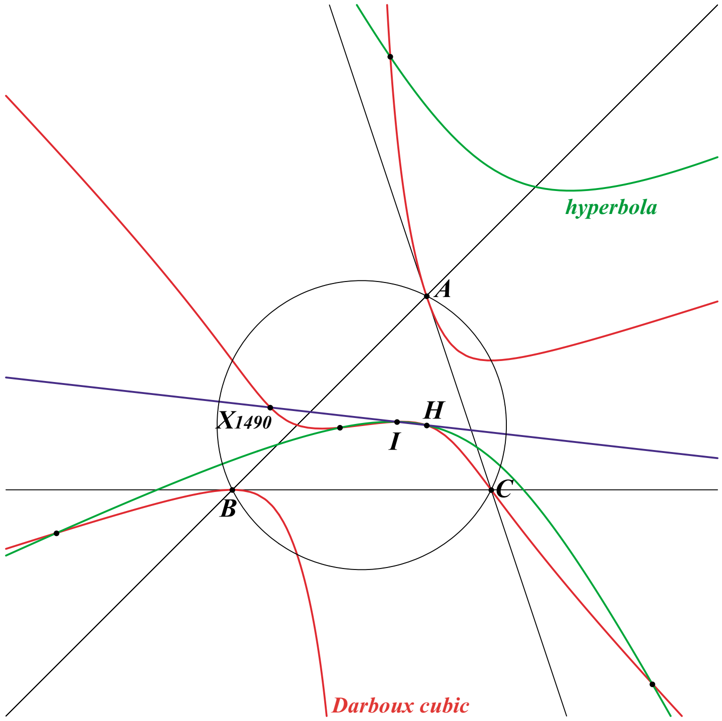

Figure 15. Proposition 14

into three lines, one always real $\mathcal{L}_P$ containing R_1 and R_2, two other passing through H.

Lemma 17. *$\mathcal{L}_P$ contains the Nagel point X_8. In other words, X_8, R_1 and R_2 are always collinear.*

With $P = (u : v : w)$, $\mathcal{L}_P$ has equation :

$$\sum_{\text{cyclic}} \frac{a(cv - bw)}{b + c - a} x = 0$$

$\mathcal{L}_P$ is the trilinear polar of the isotomic conjugate of point T, where T is the barycentric product of X_{57} and the isotomic conjugate of the trilinear pole of the line PI. The construction of R_1 and R_2 is now possible in the most general case with one of the conics and $\mathcal{L}_P$. Nevertheless, in three specific situations already mentioned, the construction simplifies as we see in the three following corollaries.

Corollary 18. *When P lies on IH, there is only one* (*always real*) *Mandart triangle $\mathbf{T}_t(P)$ perspective to ABC. The perspector R is the intersection of the lines HX_8 and PX_{78}.*

Proof. This is obvious since equation (2) is at most of the first degree when P lies on IH. □

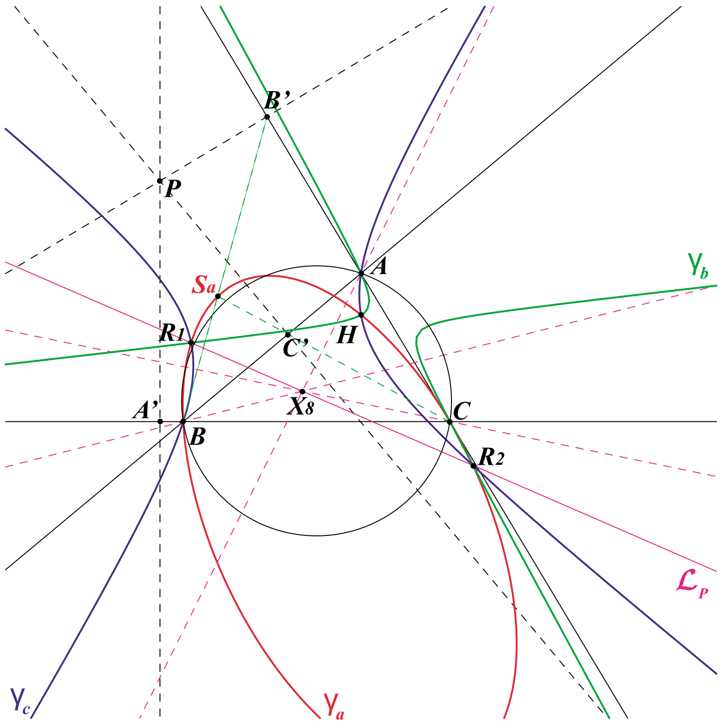

Figure 16. The three conics γ_a, γ_b, γ_c and the perspectors R_1, R_2

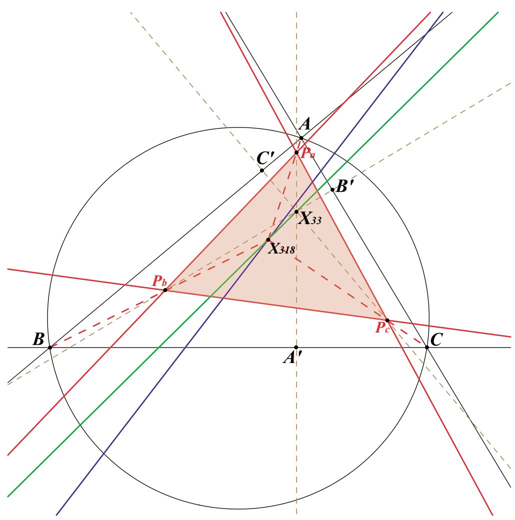

Figure 17. Only one triangle $P_aP_bP_c$ perspective to ABC when P lies on IH

In Figure 17, we have taken $P = X_{33}$ and $R = X_{318}$.

Remark. The line IH meets the Darboux cubic again at X_{1490}. The corresponding Mandart triangle $\mathbf{T}_t(P)$ is the pedal triangle of X_{1490} which is also the cevian triangle of X_{329}.

Corollary 19. *When P (different from I and H) lies on the conic seen above, there are two (not always real) Mandart triangles $\mathbf{T}_t(P)$ perspective to ABC obtained for two opposite values t_1 and t_2. The vertices of the triangles are therefore two by two symmetric in the sidelines of ABC.*

In the figure 18, we have taken $P = X_{500}$ (orthocenter of the incentral triangle).

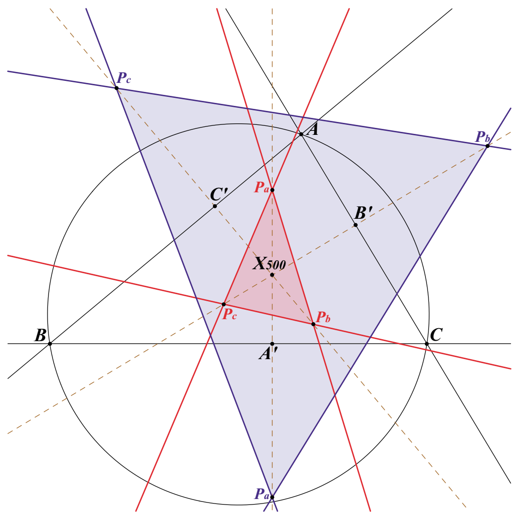

Figure 18. Two triangles $P_aP_bP_c$ perspective with ABC having vertices symmetric in the sidelines of ABC

Corollary 20. *When P (different from I, H, X_{1490}) lies on the Darboux cubic, there are two (always real) Mandart triangles $\mathbf{T}_t(P)$ perspective to ABC, one of them being the pedal triangle of P with a perspector on the Lucas cubic.*

Since one perspector, say R_1, is known, the construction of the other is simple: it is the "second" intersection of the line X_8R_1 with the conic $BCHS_aR_1$.

Table 3 gives P (on the Darboux cubic), the corresponding perspectors R_1 (on the Lucas cubic) and R_2.

Table 3

P	X_1	X_3	X_4	X_{20}	X_{40}	X_{64}	X_{84}	X_{1498}
R_1	X_7	X_2	X_4	X_{69}	X_8	X_{253}	X_{189}	X_{20}
R_2		X_8	X_4	X_{388}	X_{10}	$*$	X_{515}	$*$

Table 4

Triangle center	First barycentric coordinate
$R_2(X_{64})$	$\frac{a^8-4a^6(b+c)^2+2a^4(b+c)^2(3b^2-4bc+3c^2)-4a^2(b^2-c^2)^2(b^2+c^2)+(b-c)^2(b+c)^6}{b+c-a}$
$R_2(X_{1498})$	$\frac{a^4-2a^2(b+c)^2+(b^2-c^2)^2)}{a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2}$

In Figure 19, we have taken $P = X_{40}$ (reflection of I in O).

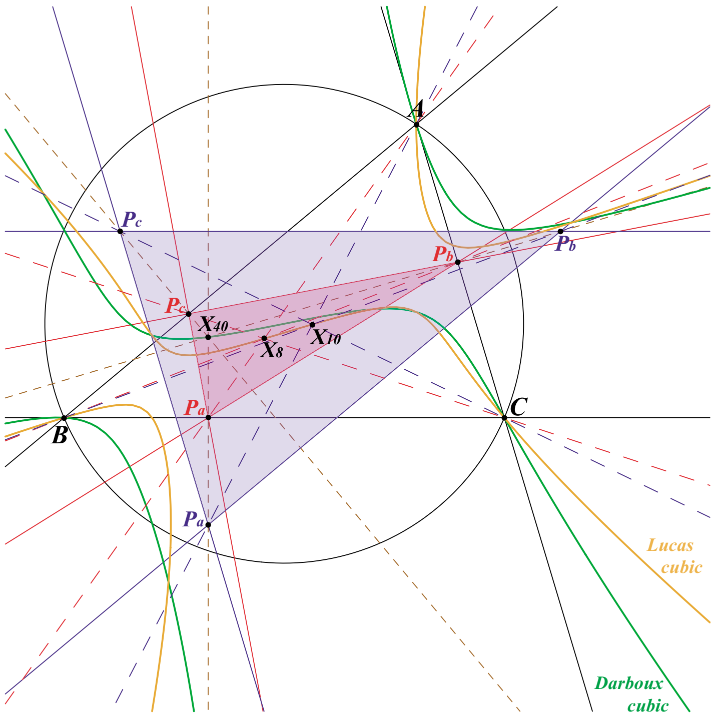

Figure 19. Two triangles $P_aP_bP_c$ perspective with ABC when $P = X_{40}$

Proposition 21. *The triangles $A'B'C'$ and $P_aP_bP_c$ have the same area if and only if*
(1) $t = 0$, or
(2) $t = -\frac{bc(b+c)u+ca(c+a)v+ab(a+b)w}{2R(a+b+c)(u+v+w)}$, [9]
(3) t is a solution of a quadratic equation[10] *whose discriminant has the same sign of*

$$f(u,v,w) = \sum_{\text{cyclic}} b^2c^2(b+c)^2u^2 + 2a^2bc(bc - 3a(a+b+c))vw.$$

[9]This can be interpreted as $t = -\frac{d(P)}{d(O)} \cdot R$, where $d(X)$ denotes the distance from X to the polar line of I in the circumcircle.

[10]$abc(a+b+c)(u+v+w)^2t^2 + 2\Delta(u+v+w)\left(\sum_{\text{cyclic}} bc(b+c)u\right)t + 8\Delta^2(a^2vw + b^2wu + c^2uv) = 0.$

The equation $f(x, y, z) = 0$ represents an ellipse $\mathcal{E}$ centered at X_{35} [11] whose axes are parallel and perpendicular to the line OI. See Figure 20.

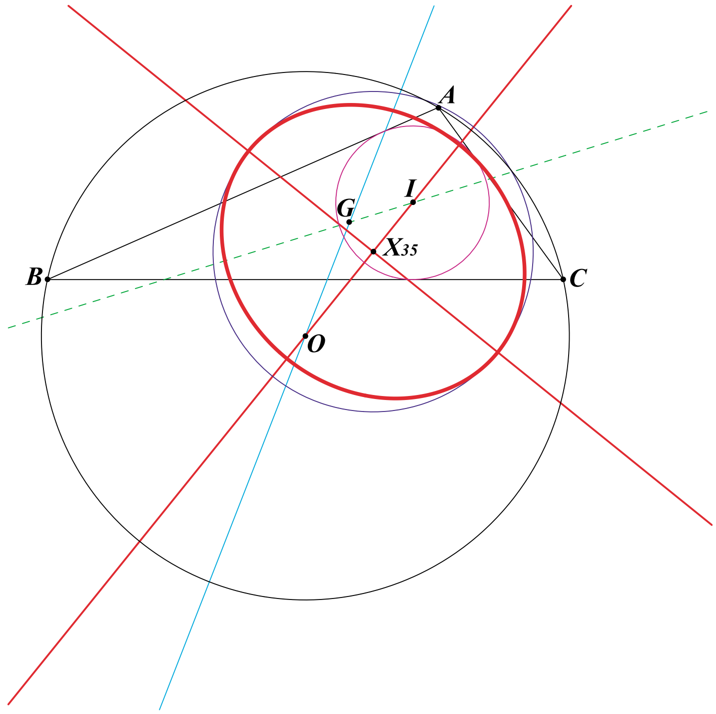

Figure 20. The "critical" ellipse $\mathcal{E}$

According to the position of P with respect to this ellipse, it is possible to have other triangles solution of the problem. More precisely, if P is
– inside $\mathcal{E}$, there is no other triangle,
– outside $\mathcal{E}$, there are two other (distinct) triangles,
– on $\mathcal{E}$, there is only one other triangle.

Proposition 22. *As t varies, each line P_bP_c, P_cP_a, P_aP_b still envelopes a parabola.*

Denote these parabolas by $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively. $\mathcal{P}_a$ has focus the projection F_a of P on AI and directrix ℓ_a parallel to AI at E_a such that $\overrightarrow{PE_a} = \cos A\ \overrightarrow{PF_a}$. Note that the direction of the directrix (and the axis) is independent of P. $\mathcal{P}_a$ is still tangent to the lines PB', PC', $B'C'$.

In this more general case, the directrices ℓ_a, ℓ_b, ℓ_c are not necessarily concurrent. This happens if and only if P lies on the line OI and, then, their common point lies on IG.

Proposition 23. *The Mandart triangle $\mathbf{T}_t(P)$ and the pedal triangle of P are perspective at P. As t varies, the envelope of their perspectrix is a parabola.*

[11] Let I'_a be the inverse-in-circumcircle of the excenter I_a, and define I'_b and I'_c similarly. The triangles ABC and $I'_aI'_bI'_c$ are perspective at X_{35} which is a point on the line OI.

The directrix of this parabola is parallel to the line IP^*. It is still inscribed in the pedal triangle $A'B'C'$ of P and is tangent to the two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ met in proposition 13.

Remark. Unlike the case $P = X_8$, ABC is not necessary self polar with respect to this Mandart parabola.

Proposition 24. *The Mandart triangle $\mathbf{T}_t(P)$ and ABC are orthologic. The perpendiculars from A, B, C to the corresponding sidelines of $P_aP_bP_c$ are concurrent at $Q = \left(\frac{a^2}{at+2\Delta u} : \cdots : \cdots\right)$. As t varies, the locus of Q is generally the circumconic which is the isogonal transform of the line IP.*

This conic has equation

$$\sum_{\text{cyclic}} a^2(cv - bw)yz = 0.$$

It is tangent at I to IP, and is a rectangular hyperbola if and only if P lies on the line OI ($P \neq I$). When $P = I$, the triangles are homothetic at I and the perpendiculars concur at I.

References

[1] Brocard H. and Lemoyne T. *Courbes Géométriques Remarquables*. Librairie Albert Blanchard, Paris, third edition, 1967.

[2] F.G.-M. *Exercices de géométrie*. 1920, Gabay reprint, 1991.

[3] Kimberling C. *Triangle Centers and Central Triangles*. Congressus Numerantium 129, Winnipeg, Canada, 1998.

[4] Kimberling C. *Encyclopedia of Triangle Centers*. http://www2.evansville.edu/ck6/encyclopedia.

[5] Mandart H. *Sur l'hyperbole de Feuerbach*, Mathesis, 1893, pp.81-89.

[6] Mandart H. *Sur une ellipse associée au triangle*, Mathesis, 1894, pp. 241-245.

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