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Triangle Centers with Linear Intercepts and Linear Subangles

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Abstract. Let ABC be a triangle with side-lengths a, b, and c, and with angles A, B, and C. Let AA', BB', and CC' be the cevians through a point V, let x, y, and z be the lengths of the segments BA', CB', and AC', and let ξ, η, and ζ be the measures of the angles $\angle BAA'$, $\angle CBB'$, and $\angle ACC'$. The centers V for which x, y, and z are linear forms in a, b, and c are characterized. So are the centers for which ξ, η, and ζ are linear forms in A, B, and C.

Let ABC be a non-degenerate triangle with side-lengths a, b, and c, and let V be a point in its plane. Let AA', BB', and CC' be the cevians of ABC through V and let the intercepts x, y, and z be defined to be the directed lengths of the segments BA', CB', and AC', where x is positive or negative according as A' and C lie on the same side or on opposite sides of B, and similarly for y and z; see Figure 1. To avoid infinite intercepts, we assume that V does not lie on any of the three exceptional lines passing through the vertices of ABC and parallel to the opposite sides.

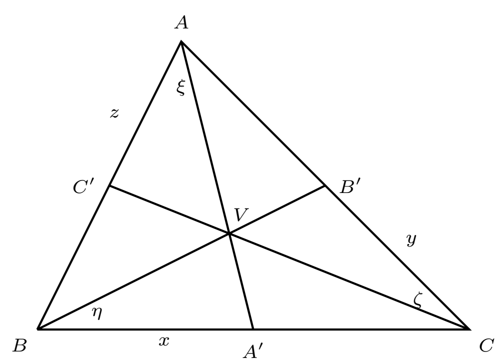

Figure 1

If V is the centroid of ABC, then the intercepts (x, y, z) are clearly given by $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$. It is also easy to see that the triples (x, y, z) determined by the Gergonne and Nagel points are

$$\left(\frac{a-b+c}{2}, \frac{a+b-c}{2}, \frac{-a+b+c}{2}\right), \left(\frac{a+b-c}{2}, \frac{-a+b+c}{2}, \frac{a-b+c}{2}\right),$$

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respectively. We now show that these are the only three centers whose corresponding intercepts (x, y, z) are linear forms in a, b, and c. Here, and in the spirit of [4] and [5], a center is a function that assigns to a triangle, in a family $\mathbf{U}$ of triangles, a point in its plane in a manner that is symmetric and that respects isometries and dilations. It is assumed that $\mathbf{U}$ has a non-empty interior, where $\mathbf{U}$ is thought of as a subset of $\mathbf{R}^3$ by identifying a triangle ABC with the point (a, b, c).

Theorem 1. *The triangle centers for which the intercepts x, y, z are linear forms in a, b, c are the centroid, the Gergonne and the Nagel points.*

Proof. Note first that if (x, y, z) are the intercepts corresponding to a center V, and if

$$x = \alpha a + \beta b + \gamma c,$$

then it follows from reflecting ABC about the perpendicular bisector of the segment BC that

$$a - x = \alpha a + \beta c + \gamma b.$$

Therefore $\alpha = \frac{1}{2}$ and $\beta + \gamma = 0$. Applying the permutation $(A\ B\ C) = (a\ b\ c) = (x\ y\ z)$, we see that

$$x = \alpha a + \beta b + \gamma c,\ y = \alpha b + \beta c + \gamma a,\ z = \alpha c + \beta a + \gamma b.$$

Substituting in the cevian condition $xyz = (a - x)(b - y)(c - z)$, we obtain the equation

$$\begin{aligned} & \left(\frac{a}{2} + \beta(b-c)\right)\left(\frac{b}{2} + \beta(c-a)\right)\left(\frac{c}{2} + \beta(a-b)\right) \\ = \ & \left(\frac{a}{2} - \beta(b-c)\right)\left(\frac{b}{2} - \beta(c-a)\right)\left(\frac{c}{2} - \beta(a-b)\right) \end{aligned}$$

which simplifies into

$$\beta\left(\beta + \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)(a-b)(b-c)(c-a) = 0.$$

This implies the three possibilities $\beta = 0, -\frac{1}{2}$, or $\frac{1}{2}$ that correspond to the centroid, the Gergonne point and the Nagel point, respectively. $\square$

In the same vein, the cevians through V define the subangles ξ, η, and ζ of the angles A, B, and C of ABC as shown in Figure 1. These are given by

$$\xi = \angle BAV, \qquad \eta = \angle CBV, \qquad \zeta = \angle ACV.$$

Here we temporaily take V to be inside ABC for simplicity, and treat the general case in Note 1 below. It is clear that the subangles (ξ, η, ζ) corresponding to the incenter of ABC are given by $(\frac{A}{2}, \frac{B}{2}, \frac{C}{2})$. Also, if ABC is acute-angled, then the orthocenter and circumcenter lie inside ABC and the triples (ξ, η, ζ) of subangles that they determine are given by

$$\left(\frac{A-B+C}{2}, \frac{A+B-C}{2}, \frac{-A+B+C}{2}\right), \left(\frac{A+B-C}{2}, \frac{-A+B+C}{2}, \frac{A-B+C}{2}\right), \quad (1)$$

or equivalently by

$$\left(\frac{\pi}{2}-B,\frac{\pi}{2}-C,\frac{\pi}{2}-A\right),\ \left(\frac{\pi}{2}-C,\frac{\pi}{2}-A,\frac{\pi}{2}-B\right), \tag{2}$$

respectively. Here again, we prove that these are the only centers whose corresponding subangles (ξ,η,ζ) are linear forms in A, B, and C. As before, we first show that the subangles (ξ,η,ζ) determined by such a center are of the form

$$\xi=\alpha A+\beta B+\gamma C,\ \eta=\alpha B+\beta C+\gamma A,\ \zeta=\alpha C+\beta A+\gamma B,$$

where $\alpha=\frac{1}{2}$ and $\beta+\gamma=0$. Substituting in the trigonometric cevian condition

$$\sin\xi\ \sin\ \eta\ \sin\ \zeta\ =\ \sin\left(a-\xi\right)\ \sin\left(b-\eta\right)\ \sin\left(c-\zeta\right), \tag{3}$$

we obtain the equation

$$\begin{aligned}&\sin\left(\frac{A}{2}+\beta(B-C)\right)\ \sin\left(\frac{B}{2}+\beta(C-A)\right)\ \sin\left(\frac{C}{2}+\beta(A-B)\right)\\ =\ &\sin\left(\frac{A}{2}-\beta(B-C)\right)\ \sin\left(\frac{B}{2}-\beta(C-A)\right)\ \sin\left(\frac{C}{2}-\beta(A-B)\right).\end{aligned} \tag{4}$$

Using the facts that

$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\ ,\ \beta(B-C)+\beta(C-A)+\beta(A-B)=0,$$

and the facts [3, Formulas 677, 678, page 166] that if $u+v+w=0$, then

$$\begin{aligned}4\ \cos\ u\ \cos\ v\ \sin\ w\ &=\ -\sin\ 2u\ -\ \sin\ 2v\ +\ \sin\ 2w,\\ 4\ \sin\ u\ \sin\ v\ \sin\ w\ &=\ -\sin\ 2u\ -\ \sin\ 2v\ -\ \sin\ 2w,\end{aligned}$$

and that if $u+v+w=\pi/2$, then

$$\begin{aligned}4\ \cos\ u\ \cos\ v\ \cos\ w\ &=\ \sin\ 2u\ +\ \sin\ 2v\ +\ \sin\ 2w,\\ 4\ \sin\ u\ \sin\ v\ \cos\ w\ &=\ \sin\ 2u\ +\ \sin\ 2v\ -\ \sin\ 2w,\end{aligned}$$

(4) simplifies into

$$\sin A\ \sin(2\beta(B-C))+\sin\ B\ \sin(2\beta(C-A))+\sin\ C\ \sin(2\beta(A-B))=0. \tag{5}$$

It is easy to check that for $\beta=-\frac{1}{2}$, 0, and $\frac{1}{2}$, this equation is satisfied for all triangles. Conversely, since (5) holds on a set **U** having a non-empty interior, it holds for all triangles, and in particular it holds for the triangle $(A,B,C)=(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{6})$. This implies that

$$\sin\frac{\beta\pi}{3}\ \left(\cos\frac{\beta\pi}{3}-\frac{\sqrt{3}}{2}\right)\ =\ 0.$$

Since $-\frac{3}{2}\le\beta\le\frac{3}{2}$ for this particular triangle, it follows that β must be $-\frac{1}{2}$, 0, or $\frac{1}{2}$. Thus the only solutions of (5) are $\beta=-\frac{1}{2}$, 0, and $\frac{1}{2}$. These correspond to the orthocenter, incenter and circumcenter, respectively. We summarize the result in the following theorem.

Theorem 2. *The triangle centers for which the subangles ξ, η, ζ are linear forms in A, B, C are the orthocenter, incenter, and circumcenter.*

Remarks. (1) Although the subangles ξ, η, and ζ of a given point V were defined for points that lie inside ABC only, it is possible to extend this definition to include exterior points also, without violating the trigonometric version (3) of Ceva's concurrence condition or the formulas (1) and (2) for the subangles corresponding to the orthocenter and the circumcenter. To do so, we let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the open half planes determined by the line that is perpendicular at A to the internal angle-bisector of A, where we take $\mathbf{H}_1$ to be the half-plane containing B and C. For $V \in \mathbf{H}_1$, we define the subangle ξ to be the signed angle $\angle BAV$, where $\angle BAV$ is taken to be positive or negative according as the rotation within $\mathbf{H}_1$ that takes AB to AV has the same or opposite handedness as the one that takes AB to AC. For $V \in \mathbf{H}_2$, we stipulate that V and its reflection about A have the same subangle ξ. We define η and ζ similarly. Points on the three exceptional lines that are perpendicular at the vertices of ABC to the respective internal angle-bisectors are excluded.

(2) In terms of the intercepts and subangles, the first (respectively, the second) Brocard point of a triangle is the point whose subangles ξ, η, and ζ satisfy $\xi = \eta = \zeta$ (respectively, $A - \xi = B - \eta = C - \zeta$.) Similarly, the first and the second Brocard-like Yff points are the points whose intercepts x, y, and z satisfy $x = y = z$ and $a - x = b - y = c - z$, respectively. Other Brocard-like points corresponding to features other than intercepts and subangles are being explored by the authors.

(3) The requirement that the intercepts x, y, and z be linear in a, b, and c is quite restrictive, since the cevian condition has to be observed. It is thus tempting to weaken this requirement, which can be written in matrix form as $[x\ y\ z] = [a\ b\ c]L$, where L is a 3×3 matrix, to take the form $[x\ y\ z]M = [a\ b\ c]L$, where M is not necessarily invertible. The family of centers defined by this weaker requirement, together of course with the cevian condition, is studied in detail in [2]. So is the family obtained by considering subangles instead of intercepts.

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